Explaining Total Factor Productivity

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"Needed: A Theory of Total Factor Productivity"

Edward C. Prescott (1998)

1. Introduction

- Total Factor Productivity (TFP) has become the choice measure of productivity
- TFP is often referred to as the Solow residual, and it is generally just that, namely a residual
- TFP is rather opaque as to the nature of the phenomena that it pertains to measure
- It is difficult to reconcile TFP with various models of factor augmenting technological change
- Is technological change neutral or is it biased?
- If it is neutral, is it neutral in the sense of Hicks, Harrod, or Solow?

1. Introduction, *continued*

- Do increases in productivity, as captured by TFP, necessarily imply increases in real wages?
- What about the real return on capital, must it necessarily increase too?
- The purpose of this paper is to sort out some of these questions…
- … and to show how TFP can be decomposed into the contribution of labor and the contribution of capital
- As an illustration, some estimates for the United States are reported

2. Index Number Approach

- Total factor productivity can be defined as the part of output growth that cannot be explained by input growth
- Notation:
  - y_t , p_t quantity and price of output
  - $x_{K,t}$, $w_{K,t}$ quantity and price of capital services
  - $x_{L,t}$, $w_{L,t}$ quantity and price of labor services

2. Index Number Approach, *continued*

A state-of-the art measure of TFP is given by the following index:

(1) $$T_{t,t-1} = \frac{Y_{t,t-1}}{X_{t,t-1}}$$

where

(2) $$Y_{t,t-1} \equiv \frac{y_t}{y_{t-1}}$$

(3) $$X_{t,t-1} \equiv \exp\left[\frac{1}{2}(s_{K,t} + s_{K,t-1})\ln\frac{x_{K,t}}{x_{K,t-1}} + \frac{1}{2}(s_{L,t} + s_{L,t-1})\ln\frac{x_{L,t}}{x_{L,t-1}}\right]$$

(4) $$s_{j,t} \equiv \frac{w_{j,t}x_{j,t}}{p_t y_t}, \quad j \in \{K, L\}$$

2. Index Number Approach, *continued*

- Using the data of Kohli (2010) for the United States, one finds that TFP has averaged about 1.09% per year over the period 1970 – 2001
- While this is useful information, it tells us nothing about the nature of technological change, and whether it benefited capital or labor, or both

3. Production function approach

- TFP can also be defined with reference to a production function
- This actually leads to for ***four*** interpretations of TFP

3. Production function approach, *continued*

Aggregate production function:

(5) $$y_t = f(x_{K,t}, x_{L,t}, t)$$

First-order conditions:

(6) $$\frac{\partial f(\cdot)}{\partial x_{K,t}} = \frac{w_{K,t}}{p_t}$$

(7) $$\frac{\partial f(\cdot)}{\partial x_{L,t}} = \frac{w_{L,t}}{p_t}$$

3. Production function approach, *continued*

Let $\mu_t \equiv \partial \ln y_t / \partial t$ be the instantaneous rate of technological change; we then have:

$$(8) \qquad \frac{\partial f(\cdot)}{\partial t} = \mu_t y_t$$

Following Diewert and Morrison (1986), we define the following index of TFP:

$$(10) \qquad T_{t,t-1} \equiv \sqrt{\frac{f(x_{K,t-1}, x_{L,t-1}, t)}{f(x_{K,t-1}, x_{L,t-1}, t-1)} \frac{f(x_{K,t}, x_{L,t}, t)}{f(x_{K,t}, x_{L,t}, t-1)}} \qquad \textit{(interpretation 1)}$$

3. Production function approach, *continued*

Assume that the production function has the Translog form:

(11)
$$\ln y_t = \alpha_0 + \beta_K \ln x_{K,t} + (1-\beta_K)\ln x_{L,t} + \frac{1}{2}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})^2 + \beta_T t + \phi_{KT}(\ln x_{K,t} - \ln x_{L,t})\, t + \frac{1}{2}\phi_{TT} t^2$$

The inverse input demand functions:

(12)
$$s_{K,t} = \frac{\partial \ln f(\cdot)}{\partial \ln x_{K,t}} = \beta_K + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t}) + \phi_{KT} t$$

(13)
$$s_{L,t} = \frac{\partial \ln f(\cdot)}{\partial \ln x_{L,t}} = (1-\beta_K) - \phi_{KK}(\ln x_{K,t} - \ln x_{L,t}) - \phi_{KT} t$$

3. Production function approach, *continued*

The instantaneous rate of technological change:

(14) $\mu_t = \dfrac{\partial \ln f(\cdot)}{\partial t} = \beta_T + \phi_{KT}(\ln x_{K,t} - \ln x_{L,t}) + \phi_{TT} t$

Introducing (11) into (10) yields the following measure of TFP:

(15) $\ln T_{t,t-1} = \beta_T + \frac{1}{2}\phi_{KT}(\ln x_{K,t} - \ln x_{L,t}) + \frac{1}{2}\phi_{KT}(\ln x_{K,t-1} - \ln x_{L,t-1}) + \frac{1}{2}\phi_{TT}(2t-1)$

A second interpretation of TFP, in view of (14):

(16) $\ln T_{t,t-1} = \frac{1}{2}(\mu_t + \mu_{t-1})$ *(interpretation 2)*

3. Production function approach, *continued*

By Diewert's (1976) quadratic approximation lemma:

(17) $\ln T_{t,t-1} = \bar{\mu}_{t,t-1}$ *(interpretation 3)*

Finally, one can show that if the production function is Translog, then *(interpretation 4)*:

(20)

$$\ln Y_{t,t-1} - \ln X_{t,t-1} = \beta_T + \frac{1}{2}\phi_{KT}(\ln x_{K,t} - \ln x_{L,t}) + \frac{1}{2}\phi_{KT}(\ln x_{K,t-1} - \ln x_{L,t-1}) + \frac{1}{2}\phi_{TT}(2t-1)$$
$$= \ln T_{t,t-1}$$

which is equivalent to (1) – (4)

3. Production function approach, *continued*

- TFP can thus be interpreted in four different ways:
- (1) it is the change in output made possible by the passage of time, holding input quantities constant
- (2) it is the average of the instantaneous rates of technological change of times t-1 and t
- (3) it is the average rate of technological change between times t-1 and t
- (4) it is the part of output growth that cannot be explained by input growth
- In the Translog case, all four interpretations are equivalent

3. Production function approach, *continued*

- Estimates of the Translog production function from Kohli (2010) are reported in Table 1, column 1
- TFP computed according to (15) – or equivalently (16), (17), or (20) – averaged 1.02% over the period 1970-2001

Table 1

Parameter estimates

	(11)
α_0	8.38851
	(3522.9)
β_K	0.27365
	(189.6)
ϕ_{KK}	-0.15322
	(-3.75)
ϕ_{KT}	0.00171
	(5.19)
β_T	0.01026
	(24.25)
ϕ_{TT}	0.00008
	(1.93)
LL	226.70
μ_{2001}	0.01158
$s_{K,2001}$	0.28452
$\psi_{KL,2001}$	1.75267

4. Impact of TFP on factor rental prices

- With the index number approach, one does not need econometric estimates of the parameters of the production function to measure TFP; that makes it very attractive
- On the other hand, this approach tells us nothing about the nature of technological change, or about its impact on income shares or on the two factor rental prices
- The econometric approach is more revealing in this respect
- The sign of ϕ_{KT} is essential in determining the impact of the passage of time on factor shares

4. Impact of TFP on factor rental prices, *continued*

- If $\phi_{KT} > 0$, as it turns out in the U.S. case, one can say that technological is pro-capital and anti-labor biased, in the sense that it increases the share of capital over time and reduces the share of labor
- Capital is thus favored at the expense of labor
- What about factor rental prices, though?
- Clearly, if technological change leads to an increase in output, for given factor endowments, and to an increase in the share of capital, it must increase the real return to capital
- But what about labor?

4. Impact of TFP on factor rental prices, *continued*

From (7) we can write the rental price of labor as follows:

$$(21) \qquad w_{L,t} = \frac{\partial f(x_{K,t}, x_{L,t}, t)}{\partial x_{L,t}} p_t = s_{L,t} \frac{f(x_{K,t}, x_{L,t}, t)}{x_{L,t}} p_t$$

Partially differentiating with respect to time and making use of (13), we get

$$(22) \qquad \frac{\partial w_t}{\partial t} = \frac{\partial s_{L,t}}{\partial t} \frac{p_t y_t}{x_{L,t}} + \frac{s_{L,t}}{x_{L,t}} \frac{\partial f(\cdot)}{\partial t} p_t = -\phi_{KT} \frac{p_t y_t}{x_{L,t}} + w_{L,t} \mu_t$$

4. Impact of TFP on factor rental prices, *continued*

Or, after having divided through by $w_{L,t}$:

(23) $$\hat{w}_{L,t} = \mu_t - \frac{\phi_{KT}}{s_{L,t}}$$

where the hat (^) indicates a relative change

4. Impact of TFP on factor rental prices, *continued*

- As long as the technology is progressing, the first term on the right hand side is positive
- If ϕ_{KT} is positive, technological change is anti-labor biased
- It might even be that $\phi_{KT}/s_{L,t} > \mu_t$, in which case technological change would be ultra anti-labor biased: technological change would then lead to an actual fall in the wage rate…
- … even though technological progress would unambiguously increase average labor productivity

4. Impact of TFP on factor rental prices, *continued*

- As it turns out for the U.S. case, $\phi_{KT}/s_{L,t} < \mu_t$; technological case is thus anti-labor biased, but not ultra anti-labor biased
- Nonetheless, the rate of increase in real wages is less than the rate of growth of TFP and of average labor productivity
- Over the entire sample period, real wages increased by about 46%, with 27% explained by technological change, the rest being explained by capital deepening
- Although the econometric approach yields much richer results than the index number approach, the fact remains that it still does not teach us much about the nature of the technological change process, or as to why technological change is anti-labor biased

5. Disembodied factor augmenting technological change

A more transparent approach is to assume that technological change is disembodied and factor augmenting:

(24) $\tilde{x}_{K,t} = \tilde{x}_K(x_{K,t}, t) = x_{K,t}\,\gamma_{K,t}$ with (26) $\gamma_{K,t} = e^{\mu_K t}$

(25) $\tilde{x}_{L,t} = \tilde{x}_L(x_{L,t}, t) = x_{L,t}\,\gamma_{L,t}$ with (27) $\gamma_{L,t} = e^{\mu_L t}$

so that:

(28) $\tilde{x}_{K,t} = \tilde{x}_K(x_{K,t}, t) = x_{K,t}\, e^{\mu_K t}$

(29) $\tilde{x}_{L,t} = \tilde{x}_L(x_{L,t}, t) = x_{L,t}\, e^{\mu_L t}$

5. Disembodied factor augmenting technological change, *continued*

Let the production function again be Translog:

(30) $$\ln y_t = \alpha_0 + \beta_K \ln \tilde{x}_{K,t} + (1-\beta_K)\ln \tilde{x}_{L,t} + \frac{1}{2}\phi_{KK}(\ln \tilde{x}_{K,t} - \ln \tilde{x}_{L,t})^2$$

Making use of (28) and (29), we get:

(31) $$\begin{aligned}\ln y_t = {} & \alpha_0 + \beta_K \ln x_{K,t} + (1-\beta_K)\ln x_{L,t} + \beta_K \mu_K t + (1-\beta_K)\mu_L t \\ & + \frac{1}{2}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})^2 + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\mu_K - \mu_L)t \\ & + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)^2 t^2\end{aligned}$$

5. Disembodied factor augmenting technological change, *continued*

The capital and labor shares are again obtained by logarithmic differentiation:

(32) $$s_{K,t} = \beta_K + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t}) + \phi_{KK}(\mu_K - \mu_L)t$$

(33) $$s_{L,t} = (1 - \beta_K) - \phi_{KK}(\ln x_{K,t} - \ln x_{L,t}) - \phi_{KK}(\mu_K - \mu_L)t$$

For the rate of technological change ($\mu \equiv \partial \ln y / \partial t$), we get:

(34) $$\mu_t = \beta_K \mu_K + (1 - \beta_K)\mu_L + \phi_{KK}(\mu_K - \mu_L)(\ln x_{K,t} - \ln x_{L,t}) + \phi_{KK}(\mu_K - \mu_L)^2 t$$

5. Disembodied factor augmenting technological change, *continued*

In view of (32) – (33), this can also be expressed as:

(35) $$\mu_t = s_{K,t}\mu_K + s_{L,t}\mu_L$$

Introducing (31) into (10), we get an expression for TFP:

(36) $$\begin{aligned}\ln T_{t,t-1} = {} & \beta_K\mu_K + (1-\beta_K)\mu_L + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)(\ln x_{K,t} - \ln x_{L,t}) + \\ & \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)(\ln x_{K,t-1} - \ln x_{L,t.-1}) + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)^2(2t-1)\end{aligned}$$

Taking account of (34), we again may write:

(37) $$\ln T_{t,t-1} = \frac{1}{2}(\mu_t + \mu_{t-1})$$

5. Disembodied factor augmenting technological change, *continued*

Moreover, in view of (35), we get a ***fifth*** interpretation of TFP:

$$(38) \qquad \ln T_{t,t-1} = \frac{1}{2}(s_{K,t} + s_{K,t-1})\mu_K + \frac{1}{2}(s_{L,t} + s_{L,t-1})\mu_L$$

Furthermore, if the true production function is given by (31), then (1) is again valid. Indeed, one can show that:

$$(41) \qquad \begin{aligned} &\ln Y_{t,t-1} - \ln X_{t,t-1} = \beta_K \mu_K + (1-\beta_K)\mu_L \\ &+ \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)[(\ln x_{K,t} - \ln x_{L,t}) + (\ln x_{K,t-1} - \ln x_{L,t-1})] \\ &+ \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)^2(2t-1) \\ &= \ln T_{t,t-1} \end{aligned}$$

5. Disembodied factor augmenting technological change, *continued*

- Estimates of (31), based on Kohli (2010), are shown in Table 1
- Technological change in the United States comes close to being Harrod neutral
- TFP, computed on the basis of (36) – or equivalently (37), (38), or (41) – averaged 1.02% per year between 1970 and 2001

Table 1

Parameter estimates

	(11)	*(31)*
α_0	8.38851	8.39259
	(3522.9)	(4874.6)
β_K	0.27365	0.27405
	(189.6)	(188.6)
ϕ_{KK}	-0.15322	-0.13598
	(-3.75)	(-3.31)
ϕ_{KT}	0.00171	
	(5.19)	
β_T	0.01026	
	(24.25)	
ϕ_{TT}	0.00008	
	(1.93)	
μ_K		0.00182
		(1.75)
μ_L		0.01337
		(27.75)
LL	226.70	224.09
μ_{2001}	0.01158	0.01008
$s_{K,2001}$	0.28452	0.28454
$\psi_{KL,2001}$	1.75267	1.66797

6. The decomposition of TFP between labor and capital

We now turn to the contributions of capital and labor to TFP. For this purpose, we can define the following indices in a way similar to (10) above:

$$(42) \quad T_{t,t-1}^{K} = \sqrt{\frac{f(x_{K,t}\gamma_{K,t}, \widetilde{x}_{L,t})}{f(x_{K,t}\gamma_{K,t-1}, \widetilde{x}_{L,t})} \frac{f(x_{K,t-1}\gamma_{K,t}, \widetilde{x}_{L,t-1})}{f(x_{K,t-1}\gamma_{K,t-1}, \widetilde{x}_{L,t-1})}}$$

$$(43) \quad T_{t,t-1}^{L} = \sqrt{\frac{f(\widetilde{x}_{K,t}, x_{L,t}\gamma_{L,t})}{f(\widetilde{x}_{K,t}, x_{L,t}\gamma_{L,t-1})} \frac{f(\widetilde{x}_{K,t-1}, x_{L,t-1}\gamma_{L,t})}{f(\widetilde{x}_{K,t-1}, x_{L,t-1}\gamma_{L,t-1})}}$$

6. The decomposition of TFP between labor and capital, *continued*

Making use of (26) – (27) and (30) in (42) – (43), we find:

$$\ln T^K_{t,t\text{-}1} = \beta_K \mu_K + \frac{1}{2}\phi_{KK}\mu_K[(\ln x_{K,t} - \ln x_{L,t}) + (\ln x_{K,t\text{-}1} - \ln x_{L,t\text{-}1})] +$$

(44)
$$\frac{1}{2}\phi_{KK}(\mu_K^2 - \mu_K\mu_L)\,(2t-1)$$

$$= \frac{1}{2}(s_{K,t} + s_{K,t\text{-}1})\mu_K$$

$$\ln T^L_{t,t\text{-}1} = (1-\beta_K)\mu_L - \frac{1}{2}\phi_{KK}\mu_L\,[(\ln x_{K,t} - \ln x_{L,t}) + (\ln x_{K,t\text{-}1} - \ln x_{L,t\text{-}1})] +$$

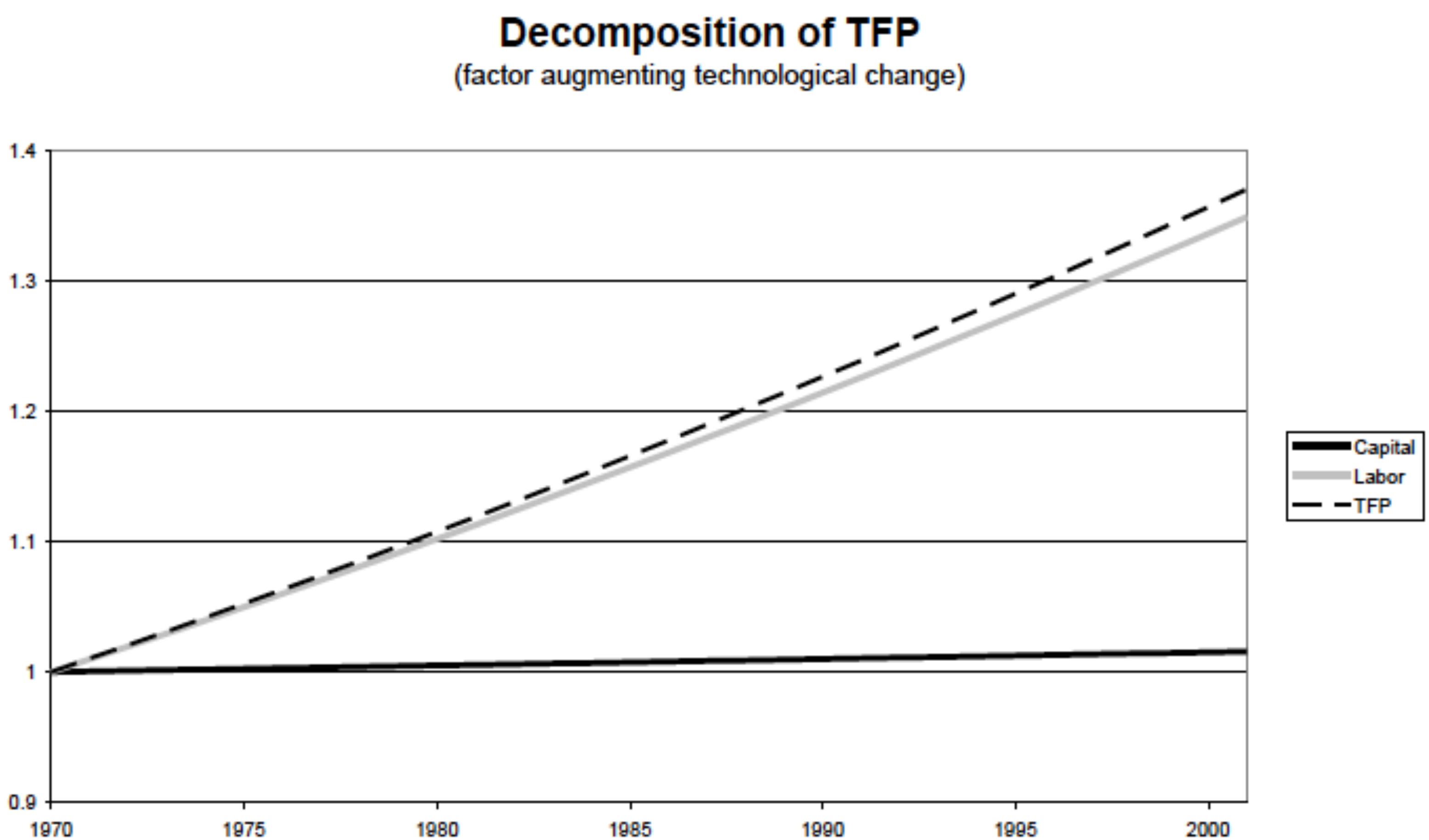
(45)
$$\frac{1}{2}\phi_{KK}(\mu_L^2 - \mu_K\mu_L)(2t-1)$$

$$= \frac{1}{2}(s_{L,t} + s_{L,t\text{-}1})\mu_L$$

so that, in view of (38):

(46) $$T_{t,t-1} = T^K_{t,t-1}T^L_{t,t-1}$$

Figure 1



Decomposition of TFP
(factor augmenting technological change)
1.4
1.3
1.2
1.1
1
0.9
1970
1975
1980
1985
1990
1995
2000
Capital
Labor
TFP

7. Factor augmenting technological change and TP flexibility

It is apparent that production function (31) contains one *less* parameter than (11). It is therefore not TP flexible. To generalize the model, let us assume now that the capital and labor efficiency factors are quadratic functions of time. We thus replace (26) and (27) by the following:

$$(47) \qquad \gamma_{K,t} = e^{\mu_K t + \frac{1}{2}\lambda_K t^2}$$

$$(48) \qquad \gamma_{L,t} = e^{\mu_L t + \frac{1}{2}\lambda_L t^2}$$

7. Factor augmenting technological change and TP flexibility, *continued*

Thus:

$$(49) \qquad \widetilde{x}_{K,t} \equiv x_{K,t} e^{\mu_K t + \frac{1}{2}\lambda_K t^2}$$

$$(50) \qquad \widetilde{x}_{L,t} \equiv x_{L,t} e^{\mu_L t + \frac{1}{2}\lambda_L t^2}$$

The instantaneous rates of factor augmentation ($\tau_{K,t}$ and $\tau_{L,t}$) are now functions of time:

$$(51) \qquad \tau_{K,t} \equiv \frac{\partial \ln \widetilde{x}_{K,t}(x_{K,t}, t)}{\partial t} = \mu_K + \lambda_K t$$

$$(52) \qquad \tau_{L,t} \equiv \frac{\partial \ln \widetilde{x}_{L,t}(x_{L,t}, t)}{\partial t} = \mu_L + \lambda_L t$$

7. Factor augmenting technological change and TP flexibility, *continued*

Introducing (49) and (50) into (30), we get:

$$\ln y_t = \alpha_0 + \beta_K \ln x_{K,t} + (1-\beta_K)\ln x_{L,t} + \beta_K \mu_K t + (1-\beta_K)\mu_L t$$
$$+ \frac{1}{2}\beta_K \lambda_K t^2 + \frac{1}{2}(1-\beta_K)\lambda_L t^2 + \frac{1}{2}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})^2$$
$$(53) \qquad + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\mu_K - \mu_L)t + \frac{1}{2}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\lambda_K - \lambda_L)t^2$$
$$+ \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)^2 t^2 + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)(\lambda_K - \lambda_L)t^3$$
$$+ \frac{1}{8}\phi_{KK}(\lambda_K - \lambda_L)^2 t^4$$

7. Factor augmenting technological change and TP flexibility, *continued*

The inverse demand functions are now as follows:

$$(54)\qquad s_{K,t} = \beta_K + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t}) + \phi_{KK}(\mu_K - \mu_L)t + \frac{1}{2}\phi_{KK}(\lambda_K - \lambda_L)t^2$$

$$(55)\qquad s_{L,t} = (1-\beta_K) - \phi_{KK}(\ln x_{K,t} - \ln x_{L,t}) - \phi_{KK}(\mu_K - \mu_L)t - \frac{1}{2}\phi_{KK}(\lambda_K - \lambda_L)t^2$$

As for the instantaneous rate of technological change, we get:

$$(56)\qquad \begin{aligned} \mu_t = {} & \beta_K\mu_K + (1-\beta_K)\mu_L + \beta_K\lambda_K t + (1-\beta_K)\lambda_L t + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\mu_K - \mu_L) \\ & + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\lambda_K - \lambda_L)t + \phi_{KK}(\mu_K - \mu_L)^2 t + \frac{3}{2}\phi_{KK}(\mu_K - \mu_L)(\lambda_K - \lambda_L)t^2 \\ & + \frac{1}{2}\phi_{KK}(\lambda_K - \lambda_L)^2 t^3 \end{aligned}$$

7. Factor augmenting technological change and TP flexibility, *continued*

In view of (51) – (52) and (54) – (55), this can be expressed as:

$$(57) \qquad \mu_t = s_{K,t}\tau_{K,t} + s_{L,t}\tau_{L,t}$$

Thus, the aggregate instantaneous rate of technological change is again found to be a weighted mean of the instantaneous rates of factor augmentation.

7. Factor augmenting technological change and TP flexibility, *continued*

Introducing (53) into (10), we find:

$$
\begin{aligned}
(58)\qquad \ln T_{t,t-1} = {} & \beta_K \mu_K + (1-\beta_K)\mu_L + \frac{1}{2}\beta_K \lambda_K (2t-1) + \frac{1}{2}(1-\beta_K)\lambda_L (2t-1) \\
& + \frac{1}{2}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\mu_K - \mu_L) + \frac{1}{4}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\lambda_K - \lambda_L)(2t-1) \\
& + \frac{1}{2}\phi_{KK}(\ln x_{K,t-1} - \ln x_{L,t-1})(\mu_K - \mu_L) \\
& + \frac{1}{4}\phi_{KK}(\ln x_{K,t-1} - \ln x_{L,t-1})(\lambda_K - \lambda_L)(2t-1) \\
& + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)^2(2t-1) + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)(\lambda_K - \lambda_L)(3t^2 - 3t + 1) \\
& + \frac{1}{8}\phi_{KK}(\lambda_K - \lambda_L)^2(4t^3 - 6t^2 + 4t - 1)
\end{aligned}
$$

7. Factor augmenting technological change and TP flexibility, *continued*

whereas in lieu of (44) and (45) we get:

(59)
$$\begin{aligned}\ln T_{t,t-1}^{K} = {} & \beta_K\mu_K + \frac{1}{2}\beta_K\lambda_K(2t-1) + \frac{1}{2}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\mu_K \\ & + \frac{1}{4}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\lambda_K(2t-1) \\ & + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)\mu_K(2t-1) \\ & + \frac{1}{4}\phi_{KK}(\mu_K - \mu_L)\lambda_K(4t^2 - 4t + 1) + \frac{1}{4}\phi_{KK}(\lambda_K - \lambda_L)\mu_K(2t^2 - 2t + 1) \\ & + \frac{1}{8}\phi_{KK}(\lambda_K - \lambda_L)\lambda_K(4t^3 - 6t^2 + 4t - 1)\end{aligned}$$

(60)
$$\begin{aligned}\ln T_{t,t-1}^{L} = {} & (1-\beta_K)\mu_L + \frac{1}{2}(1-\beta_K)\lambda_L(2t-1) \\ & - \frac{1}{2}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\mu_L \\ & - \frac{1}{4}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\lambda_L(2t-1) \\ & - \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)\mu_L(2t-1) \\ & - \frac{1}{4}\phi_{KK}(\mu_K - \mu_L)\lambda_L(4t^2 - 4t + 1) - \frac{1}{4}\phi_{KK}(\lambda_K - \lambda_L)\mu_L(2t^2 - 2t + 1) \\ & - \frac{1}{8}\phi_{KK}(\lambda_K - \lambda_L)\lambda_L(4t^3 - 6t^2 + 4t - 1)\end{aligned}$$

We again find that:

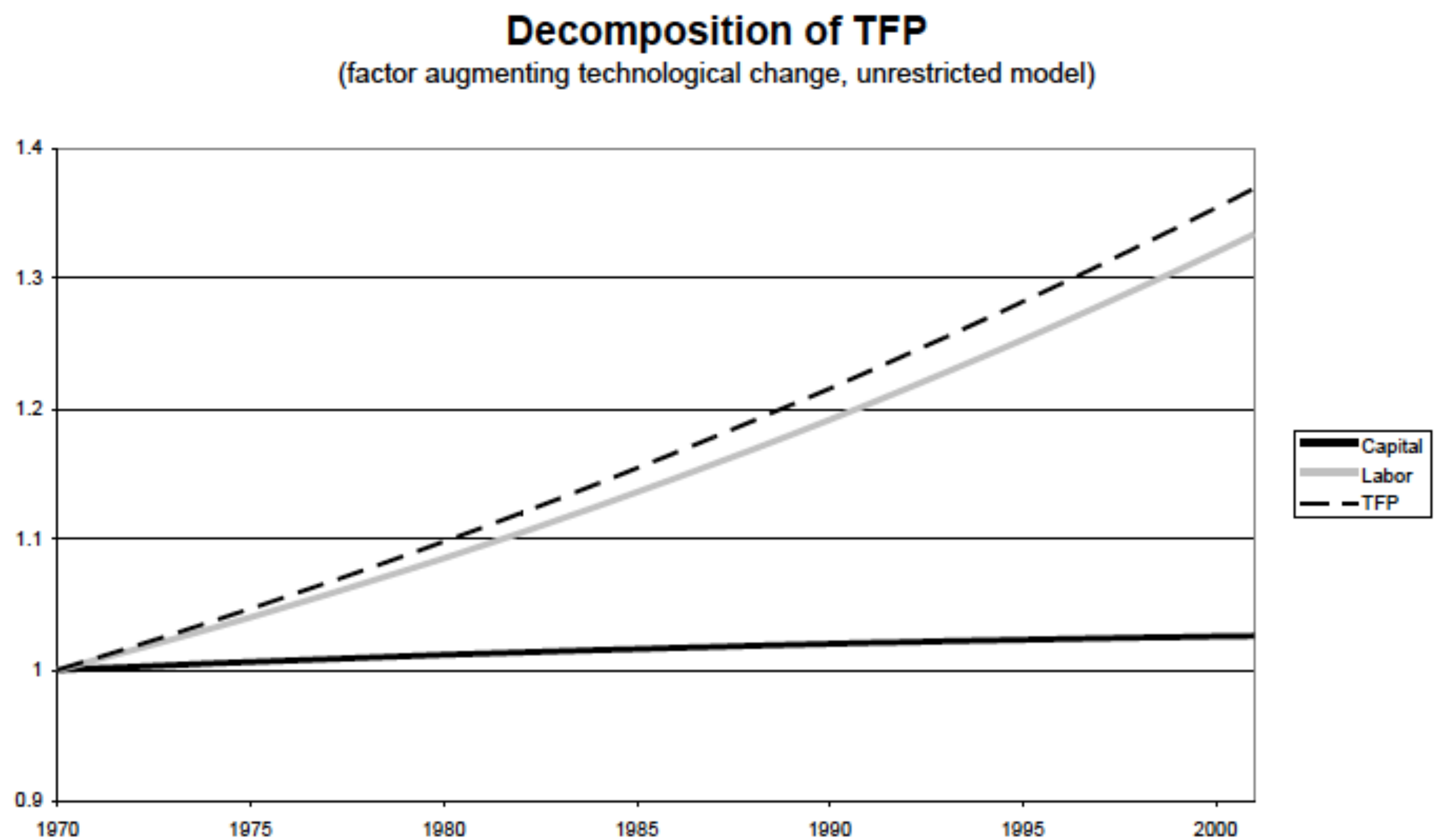
(61) $$T_{t,t-1} = T_{t,t-1}^{K} T_{t,t-1}^{L}$$

Table 1

Parameter estimates

	(11)	(31)	(53)
α_0	8.38851	8.39259	8.38944
	(3522.9)	(4874.6)	(3417.6)
β_K	0.27365	0.27405	0.27464
	(189.6)	(188.6)	(181.0)
ϕ_{KK}	-0.15322	-0.13598	-0.20418
	(-3.75)	(-3.31)	(-4.16)
ϕ_{KT}	0.00171		
	(5.19)		
β_T	0.01026		
	(24.25)		
ϕ_{TT}	0.00008		
	(1.93)		
μ_K		0.00182	0.00291
		(1.75)	(4.93)
μ_L		0.01337	0.01302
		(27.75)	(39.59)
λ_K			-0.00011
			(-0.95)
λ_L			0.00015
			(3.23)
LL	226.70	224.09	227.89
μ_{2001}	0.01158	0.01008	0.01116
$s_{K,2001}$	0.28452	0.28454	0.28007
$\psi_{KL,2001}$	1.75267	1.66797	2.01263

Figure 2


Decomposition of TFP
(factor augmenting technological change, unrestricted model)
1.4
1.3
1.2
1.1
1
0.9
1970
1975
1980
1985
1990
1995
2000
Capital
Labor
TFP

8. A parsimonious and yet flexible model

Note that (53) contains one *more* parameter than (11), i.e. one more parameter than necessary for it to be TP flexible.

A more parsimonious model is obtained by imposing the constraint $\lambda_K = \lambda_L (= \lambda)$. In that case the production function becomes:

$$\begin{aligned} \ln y_t = {} & \alpha_0 + \beta_K \ln x_{K,t} + (1-\beta_K)\ln x_{L,t} + \frac{1}{2}\phi_{KK}(\ln x_{K,t} - \ln x_{L,t})^2 \\ & + \beta_K \mu_K t + (1-\beta_K)\mu_L t + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\mu_K - \mu_L)t \\ & + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)^2 t^2 + \frac{1}{2}\lambda t^2 \end{aligned} \tag{62}$$

whereas (54) and (55) become identical to (32) and (33), and the instantaneous rate of technological change then is:

$$\mu_t = \beta_K \mu_K + (1-\beta_K)\mu_L + \phi_{KK}(\ln x_{K,t} - \ln x_{L,t})(\mu_K - \mu_L) + [\phi_{KK}(\mu_K - \mu_L)^2 + \lambda]t \tag{63}$$

8. A parsimonious and yet flexible model, *continued*

It turns out that the model of equation (62) is equivalent to (11), since there is a one to one correspondence between the two formulations, with:

(64) $\beta_T = \beta_K \mu_K + (1-\beta_K)\mu_L$

(65) $\phi_{KT} = \phi_{KK}(\mu_K - \mu_L)$

(66) $\phi_{TT} = \phi_{KK}(\mu_K - \mu_L)^2 + \lambda$

or, expressing the parameters of (62) in terms of those of (11):

(67) $\mu_K = \beta_T + (1-\beta_K)\dfrac{\phi_{KT}}{\phi_{KK}}$

(68) $\mu_L = \beta_T - \beta_K \dfrac{\phi_{KT}}{\phi_{KK}}$

(69) $\lambda = \phi_{TT} - \dfrac{\phi_{KT}^2}{\phi_{KK}}$

8. A parsimonious and yet flexible model, *continued*

For TFP we now get:

$$\ln T_{t,t-1} = \beta_K \mu_K + (1-\beta_K)\mu_L + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)(\ln x_{K,t} - \ln x_{L,t}) + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)(\ln x_{K,t-1} - \ln x_{L,t-1}) + \frac{1}{2}[\phi_{KK}(\mu_K - \mu_L)^2 + \lambda](2t-1) \quad (70)$$

This estimate of TFP is numerically identical to (15).
The contributions of capital and labor are obtained by setting $\lambda_K = \lambda_L (= \lambda)$ in (59) and (60):

$$\begin{aligned} \ln T^K_{t,t-1} = {} & \beta_K \mu_K + \frac{1}{2}\beta_K \lambda(2t-1) + \frac{1}{2}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\mu_K \\ & + \frac{1}{4}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\lambda(2t-1) \\ & + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)\mu_K(2t-1) + \frac{1}{4}\phi_{KK}(\mu_K - \mu_L)\lambda(3t^2 - 3t + 1) \end{aligned} \quad (71)$$

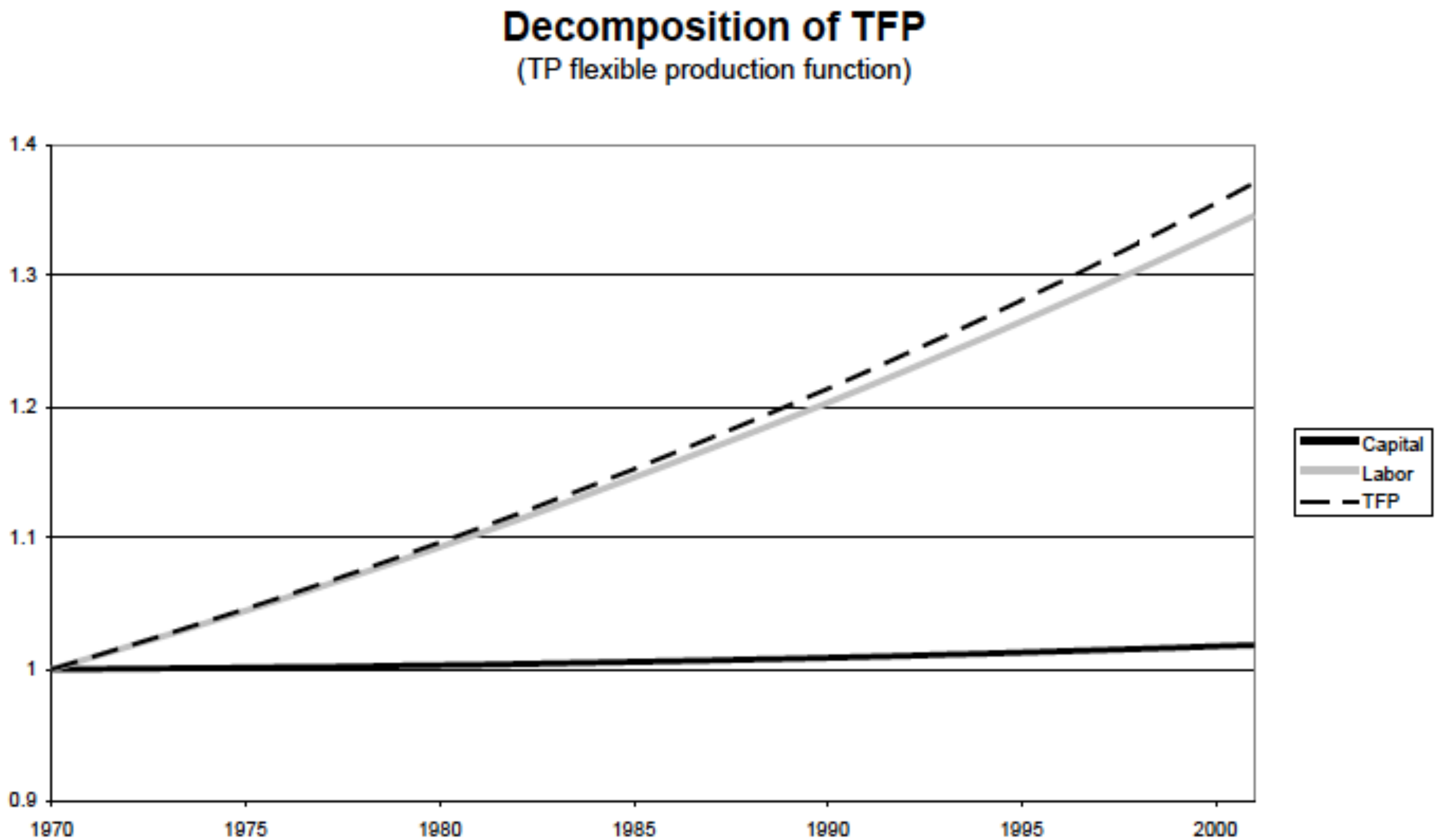
$$\begin{aligned} \ln T^L_{t,t-1} = {} & (1-\beta_K)\mu_L + \frac{1}{2}(1-\beta_K)\lambda(2t-1) \\ & + \frac{1}{2}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\mu_L \\ & + \frac{1}{4}\phi_{KK}(\ln x_{K,t} + \ln x_{K,t-1} - \ln x_{L,t} - \ln x_{L,t-1})\lambda(2t-1) \\ & + \frac{1}{2}\phi_{KK}(\mu_K - \mu_L)\mu_L(2t-1) + \frac{1}{4}\phi_{KK}(\mu_K - \mu_L)\lambda(3t^2 - 3t + 1) \end{aligned} \quad (72)$$

Table 1

Parameter estimates

	(11)	*(31)*	*(53)*	*(62)*
α_0	8.38851	8.39259	8.38944	8.38851
	(3522.9)	(4874.6)	(3417.6)	(3522.9)
β_K	0.27365	0.27405	0.27464	0.27365
	(189.6)	(188.6)	(181.0)	(189.6)
ϕ_{KK}	-0.15322	-0.13598	-0.20418	-0.15322
	(-3.75)	(-3.31)	(-4.16)	(-3.75)
ϕ_{KT}	0.00171			
	(5.19)			
β_T	0.01026			
	(24.25)			
ϕ_{TT}	0.00008			
	(1.93)			
μ_K		0.00182	0.00291	0.00217
		(1.75)	(4.93)	(2.57)
μ_L		0.01337	0.01302	0.01331
		(27.75)	(39.59)	(31.96)
λ				0.00010
				(2.38)
λ_K			-0.00011	
			(-0.95)	
λ_L			0.00015	
			(3.23)	
LL	226.70	224.09	227.89	226.70
μ_{2001}	0.01158	0.01008	0.01116	0.01158
$s_{K,2001}$	0.28452	0.28454	0.28007	0.28452
$\psi_{KL,2001}$	1.75267	1.66797	2.01263	1.75267

Figure 3



Decomposition of TFP
(TP flexible production function)
1.4
1.3
1.2
1.1
1
0.9
1970
1975
1980
1985
1990
1995
2000
Capital
Labor
TFP

9. The impact of technological change on factor rental prices reexamined, *continued*

- We can now explain why technological change is anti-labor biased in the case of the United States
- As shown by (22), technological progress must increase the real return of at least one factor, but not necessarily of both
- Take the extreme case of Harrod-neutral technological progress, which is a reasonable approximation for the United States; in that case, technological progress leads to an increase in the endowment of labor measured in efficiency units
- Output necessarily increases, and so does output per unit of labor (average labor productivity)
- The return to capital must increase as well since in the two-input case, the two inputs are necessarily Hicksian complements for each other

9. The impact of technological change on factor rental prices reexamined, *continued*

- The return to labor per efficiency unit must necessarily decrease because of diminishing marginal returns; by how much depends on the size of the elasticity of complementarity
- If capital and labor are strong Hicksian complements, the return to labor per efficiency unit will fall by a large amount, so that the return to labor per observed unit may decline, even though each unit of labor has become more efficient!

9. The impact of technological change on factor rental prices reexamined, *continued*

Let ε_{ij} be the inverse price elasticity of factor demand:

$$(73) \qquad \varepsilon_{ij} \equiv \frac{\partial \ln \tilde{w}_i(\tilde{x}_K, \tilde{x}_L, p)}{\partial \ln \tilde{x}_j}, \quad i, j \in \{K, L\}$$

Linear homogeneity of the production function implies:

$$(74) \qquad \varepsilon_{iK} + \varepsilon_{iL} = 0\,, \quad i \in \{K, L\}$$

It is well known that:

$$(75) \qquad \varepsilon_{KL} = \psi_{KL} s_L$$

$$(76) \qquad \varepsilon_{LK} = \psi_{KL} s_K$$

ψ_{KL} is the Hicksian elasticity of complementarity between capital and labor:

$$(77) \qquad \psi_{KL} \equiv \frac{f(\cdot)\dfrac{\partial^2 f(\cdot)}{\partial x_K \partial x_L}}{\dfrac{\partial f(\cdot)}{\partial x_K}\dfrac{\partial f(\cdot)}{\partial x_L}}$$

In the Translog case, it can be computed as:

$$(78) \qquad \psi_{KL} = \frac{-\phi_{KK} + s_K(1-s_K)}{s_K(1-s_K)}$$

9. The impact of technological change on factor rental prices reexamined, *continued*

We thus get for the total change in the rental price of an efficiency unit of capital:

(79) $$\hat{\tilde{w}}_K = \varepsilon_{KK}\hat{\tilde{x}}_K + \varepsilon_{KL}\hat{\tilde{x}}_L = \varepsilon_{KL}(\mu_L - \mu_K) = \psi_{KL}s_L(\mu_L - \mu_K)$$

where the hats (^) again indicate relative changes. In terms of observed factor prices:

(80) $$\hat{w}_K = \hat{\tilde{w}}_K + \mu_K + \lambda = \psi_{KL}s_L(\mu_L - \mu_K) + \mu_K + \lambda$$

and similarly for labor:

(81) $$\hat{\tilde{w}}_L = \varepsilon_{LL}\hat{\tilde{x}}_L + \varepsilon_{LK}\hat{\tilde{x}}_K = \varepsilon_{LK}(\mu_K - \mu_L) = \psi_{KL}s_K(\mu_K - \mu_L)$$

(82) $$\hat{w}_L = \hat{\tilde{w}}_L + \mu_L + \lambda = \psi_{KL}s_K(\mu_K - \mu_L) + \mu_L + \lambda = \psi_{KL}s_K\mu_K + \lambda + (1 - \psi_{KL}s_K)\mu_L$$

9. The impact of technological change on factor rental prices reexamined, *continued*

- Looking at the results for the United States, it is clear that technological progress leads to an increase in the return to capital since all three right-hand-side terms in (80) are positive
- For labor the first two terms of (82) are positive, although they are close to zero given that technological change turns out to be almost Harrod-neutral and that λ is numerically small; the third term is positive as long as $\psi_{KL} < 1/s_K$, which indeed turns out to be the case
- So we can conclude that technological progress also increases the return to labor in the U.S. case; note, however, that because the share of labor declines, the increase in real wages is less that the increase in average labor productivity, or of TFP for that matter

9. The impact of technological change on factor rental prices reexamined, *continued*

- Technological change in the United States is anti-labor biased because it is mostly labor augmenting, and because the Hicksian elasticity of complementarity between capital and labor is greater than one; these two findings together explain why technological change has a negative impact on the share of labor
- This could not have been inferred from the mere finding that ϕ_{KT} is positive: technological change would also be anti-labor biased if it were Solow neutral and if the elasticity of complementarity were less than one

10. Generalization to an arbitrary number of inputs

Assume now that there are J inputs. Taking the linear homogeneity restrictions into account, the TP-flexible Translog production function can be written as follows:

$$
\begin{aligned}
\ln y_t = \alpha_0 + \ln x_{J,t} + \sum_{j=1}^{J-1} \beta_j (\ln x_{j,t} - \ln x_{J,t}) + \beta_T t \\
(83) \qquad + \frac{1}{2} \sum_{j=1}^{J-1} \sum_{k=1}^{J-1} \phi_{jk} (\ln x_{j,t} - \ln x_{J,t})(\ln x_{k,t} - \ln x_{J,t}) \\
+ \sum_{j=1}^{J-1} \phi_{jT} (\ln x_{j,t} - \ln x_{J,t}) t + \frac{1}{2} \phi_{TT} t^2
\end{aligned}
$$

10. Generalization to an arbitrary number of inputs, *continued*

The same function expressed in terms of disembodied factor augmenting technological change becomes:

(84)

$$\begin{aligned}\ln y_t = \alpha_0 + \ln x_{J,t} + \mu_J t + \frac{1}{2}\lambda t^2 + \sum_{j=1}^{J-1}\beta_j(\ln x_{j,t} - \ln x_{J,t}) + \sum_{j=1}^{J-1}\beta_j(\mu_j - \mu_J)t \\ + \frac{1}{2}\sum_{j=1}^{J-1}\sum_{k=1}^{J-1}\phi_{jk}(\ln x_{j,t} - \ln x_{J,t})(\ln x_{k,t} - \ln x_{J,t}) \\ + \sum_{j=1}^{J-1}\sum_{k=1}^{J-1}\phi_{jk}(\ln x_{j,t} - \ln x_{J,t})(\mu_k - \mu_J)t \\ + \frac{1}{2}\sum_{j=1}^{J-1}\sum_{k=1}^{J-1}\phi_{jk}(\mu_j - \mu_J)(\mu_k - \mu_J)t^2\end{aligned}$$

10. Generalization to an arbitrary number of inputs, *continued*

We thus find:

(85) $$\mu_J + \sum_{j=1}^{J-1} \beta_j (\mu_j - \mu_J) = \beta_T$$

(86) $$\frac{1}{2} \sum_{k=1}^{J-1} \phi_{jk} (\mu_k - \mu_J) = \phi_{jT} \, , \quad j = 1, \ldots, J-1$$

(87) $$\lambda + \sum_{j=1}^{J-1} \sum_{k=1}^{J-1} \phi_{jk} (\mu_j - \mu_J)(\mu_k - \mu_J) = \phi_{TT}$$

As long as the $(J-1)\times(J-1)$ matrix of the ϕ_{jk}'s is not singular, (86) can be solved for the $J-1$ factor augmenting differentials $\mu_j - \mu_J$; μ_J, and thus all the μ_j's, can then be obtained from (85), and λ by (87).

11. Conclusions

- In this paper we attempted to explain TFP in terms of disembodied, factor augmenting technological change
- This led us to come up with **five different** interpretations of TFP:
  - (1) it is the part of output growth that cannot be explained by input growth
  - (2) it is the change in output made possible by the passage of time, holding input quantities constant
  - (3) it is the average of the instantaneous rates of technological change of times *t*-1 and *t*
  - (4) it is the average rate of technological change between times *t*-1 and *t*
  - (5) it is a moving geometric mean of the rates of factor efficiency augmentation
- In the Translog case, all five interpretations are equivalent

11. Conclusions, *continued*

- We have shown that in the case of a TP-flexible Translog production function TFP can always be interpreted as the outcome of disembodied, factor augmenting technological change
- Indeed, we have proposed a convenient way to derive the factor-augmenting rates of technological change from the estimates of such a Translog production function
- We have found that technological change is almost Harrod-neutral in the case of the United States, so that TFP is overwhelmingly explained by labor

11. Conclusions, *continued*

- Furthermore, technological change is anti-labor biased, in the sense that it tends to decrease the income share of labor; this is due to the relatively large Hicksian elasticity of complementarity between capital and labor
- Nonetheless, technological change has a positive effect on the return of both capital and labor, although the benefit to labor is less than what TFP or average labor productivity would suggest

Thank you for your attention!

Growth factors 1970-2001

Quantity of capital services:	X_K	2.25706
Quantity of labor services:	X_L	1.70513
Price of output:	P	3.76623
Quantity of output:	Y	2.52563
Price of labor services:	W_L	5.50201
Total factor productivity:	T	1.37071
Capital component of TFP:	T_K	1.01850
Labor component of TFP:	T_L	1.34581
Capital efficiency:	Γ_K	1.06789
Labor efficiency:	Γ_L	1.50832
Labor share:	S_L	0.98628
Output per unit of labor:	$A \equiv Y/X_L$	1.48119
Real wage rate:	$M \equiv W_L/P$	1.46088
Relative capital intensity:	$X \equiv X_K/X_L$	1.32369
Relative efficiency factor:	$\Gamma \equiv \Gamma_K/\Gamma_L$	0.70800
Capital intensity in efficiency terms:	$K \equiv X \cdot \Gamma$	0.93718

Approximate mean levels:

Share of capital:	s_K	0.28
Inverse price elasticity:	ε_{LK}	0.49

* * *

Decomposition of productivity growth 1970-2001:

Average labor productivity	$\boldsymbol{A = T_K \cdot T_L \cdot X^{s_K} = 1.48119}$
Capital productivity effect (4.67%)	$T_K = 1.01850$
Labor productivity effect (75.60%)	$T_L = 1.34581$
Capital deepening effect (19.73%)	$X^{s_K} \cong 1.32369^{0.28} \cong 1.08060$

Marginal labor productivity	$\boldsymbol{M = X^{\varepsilon_{LK}} \cdot \Gamma^{\varepsilon_{LK}} \cdot \Gamma_L = 1.46088}$
Capital deepening effect (36.43%)	$X^{\varepsilon_{LK}} \cong 1.32369^{0.49} \cong 1.14809$
Relative efficiency effect (-44.86%)	$\Gamma^{\varepsilon_{LK}} \cong 0.70800^{0.49} \cong 0.84362$
Factor augmentation effect (108.43%)	$\Gamma_L = 1.50832$