The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

EXACT VALUES OF TRIGONOMETRIC FUNCTIONS

For certain values of θ, the trigonometric functions $\cos\theta$, $\sin\theta$ and $\tan\theta$ have values which are easily expressed, for example, as fractions or surds. You need to know **all** of the following, without the assistance of a calculator.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0

Comments.

- The value of $\tan\theta$ is undefined when $\theta = \frac{\pi}{2}$. Please **do not** write "$\tan\frac{\pi}{2} = \infty$": this is nonsense, because ∞ is not a number.
- In university level mathematics, the *only* sensible way to measure angles is in radians. If you are in the habit of saying "$\cos 60^\circ = \frac{1}{2}$", you need to learn the radian version, otherwise the time you take for trig problems will be hugely increased. And don't forget that "$\cos 60 = \frac{1}{2}$" is not just inferior, it is **wrong**.

The above table gives (mostly) the cases when θ is an angle in the first quadrant. To evaluate trigonometric functions for angles in other quadrants you need the following formulae – for more details see the "Trigonometric identities" worksheet:

$$\cos(\theta \pm 2\pi) = \cos\theta\ ,\quad \cos(\theta \pm \pi) = -\cos\theta\ ,\quad \cos(-\theta) = \cos\theta\ ,$$
$$\sin(\theta \pm 2\pi) = \sin\theta\ ,\quad \sin(\theta \pm \pi) = -\sin\theta\ ,\quad \sin(-\theta) = -\sin\theta\ ,$$
$$\tan(\theta \pm \pi) = \tan\theta\ ,\quad \tan(-\theta) = -\tan\theta\ .$$

Also frequently useful are

$$\cos(\pi - \theta) = -\cos\theta\ ,\quad \sin(\pi - \theta) = \sin\theta\ .$$

Examples.

- $\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
- $\tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$.
- $\sin\left(\frac{5}{6}\pi\right) = \sin\left(\pi - \frac{5}{6}\pi\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.
- $\tan\left(\frac{7}{6}\pi\right) = \tan\left(\frac{7}{6}\pi - \pi\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.
- To calculate $\sin\left(\frac{100}{3}\pi\right)$, we begin by writing the coefficient as a mixed number,
$$\frac{100}{3} = 33 + \frac{1}{3}\ .$$
Then we have
$$\begin{aligned}\sin\left(\tfrac{100}{3}\pi\right) &= \sin\left(33 + \tfrac{1}{3}\right)\pi = \sin\left[\left(33 + \tfrac{1}{3}\right)\pi - 34\pi\right]\\ &= \sin\left(-\tfrac{2}{3}\pi\right) = -\sin\left(-\tfrac{2}{3}\pi + \pi\right)\\ &= -\sin\left(\tfrac{1}{3}\pi\right) = -\tfrac{\sqrt{3}}{2}\ .\end{aligned}$$
Note carefully that in the second step we subtracted an *even* multiple of π from the angle. You will often need to do this kind of calculation when studying complex numbers.

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Write out the table of exact values on page 1 from memory.
2. Calculate exact values of the following:
   (a) $\cos\left(\frac{3}{4}\pi\right)$;
   (b) $\cos\left(-\frac{5}{6}\pi\right)$;
   (c) $\tan\left(-\frac{\pi}{4}\right)$;
   (d) $\cos\left(\frac{5}{3}\pi\right)$;
   (e) $\sin\left(-\frac{\pi}{3}\right)$;
   (f) $\cos\left(-\frac{7}{4}\pi\right)$;
   (g) $\tan\left(\frac{5}{6}\pi\right)$;
   (h) $\cos(11\pi)$;
   (i) $\tan\left(\frac{125}{4}\pi\right)$.
3. Other exact values can be computed by using the addition formulae for cos, sin and tan (see the "Trigonometric identities" revision worksheet if you don't remember them).
   (a) By substituting $\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{4}$ in the formula

   $$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\ ,$$

   find the exact value of $\cos\frac{\pi}{12}$.
   (b) Use a similar method to find $\sin\frac{\pi}{12}$. Hence find $\tan\frac{\pi}{12}$.

ANSWERS.

2. (a) $-\frac{1}{\sqrt{2}}$;
   (b) $-\frac{\sqrt{3}}{2}$;
   (c) -1;
   (d) $\frac{1}{2}$;
   (e) $-\frac{\sqrt{3}}{2}$;
   (f) $\frac{1}{\sqrt{2}}$;
   (g) $-\frac{1}{\sqrt{3}}$;
   (h) -1;
   (i) 1.
3. The exact values are

   $$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}\ , \quad \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}\ ,$$

   and so

   $$\tan\left(\frac{\pi}{12}\right) = \frac{\sin\left(\frac{\pi}{12}\right)}{\cos\left(\frac{\pi}{12}\right)} = \frac{\sqrt{3}-1}{\sqrt{3}+1}\ .$$

   The last can be simplified to give

   $$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}\ :$$

   if you don't know how to do this, please see the "Surds" worksheet.