The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

TRIGONOMETRIC IDENTITIES

In order to work effectively with trigonometric functions, you need to know **all** of the following basic identities.

- Periodicity:

$$\cos\theta = \cos(\theta + 2\pi) , \qquad \sin\theta = \sin(\theta + 2\pi) .$$

- Symmetry (even and odd functions):

$$\cos(-\theta) = \cos\theta , \qquad \sin(-\theta) = -\sin\theta .$$

- Pythagoras' theorem (trigonometric version):

$$\cos^2\theta + \sin^2\theta = 1 .$$

- Other functions:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta} ,$$

  provided $\theta \neq (n + \frac{1}{2})\pi$ with n an integer;

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta} ,$$

  provided $\theta \neq n\pi$ with n an integer.

- Addition formulae:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta .\end{aligned}$$

- Double angle formulae:

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \\ \sin 2\alpha &= 2\sin\alpha\cos\alpha .\end{aligned}$$

- Exact values:

$$\cos 0 = 1 , \quad \cos\left(\frac{\pi}{2}\right) = 0 , \quad \cos\pi = -1 ,$$

$$\sin 0 = 0 , \quad \sin\left(\frac{\pi}{2}\right) = 1 , \quad \sin\pi = 0 .$$

  You should also know the values of $\cos\left(\frac{\pi}{6}\right)$, $\sin\left(\frac{\pi}{4}\right)$ and so on. See a separate revision worksheet for these.

You also need to be able to use the above formulae to derive others. For example, take the addition formula for cos, replace β by $-\beta$, and use the fact that cos is even and sin is odd:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta .\end{aligned}$$

Now add these two equations,

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta , \qquad (*)$$

to obtain a formula for a product of cosines,

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2} .$$

Remember, you **do not** need to memorise this, you need to know how to work it out.

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Use the formulae on pages 1–2 (and other basic algebraic identities) to prove that
   (a) $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$;
   (b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$;
   (c) $\sin^2(\alpha+\beta) - \sin^2(\alpha-\beta) = \sin(2\alpha)\sin(2\beta)$.
2. Divide both sides of $\cos^2\theta + \sin^2\theta = 1$ by $\cos^2\theta$ and hence find a relation between $\tan\theta$ and $\sec\theta$.
3. Prove that for any θ we have
$$\sin(\pi-\theta) = \sin\theta\ , \qquad \cos(\pi-\theta) = -\cos\theta\ ;$$
$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\ , \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\ .$$
4. Find a formula for $\tan(\alpha+\beta)$ by means of the following procedure: write down the definition of $\tan(\alpha+\beta)$; use addition formulae to expand the numerator and denominator; divide numerator and denominator by $\cos\alpha\cos\beta$; simplify.
5. *"Products–to–sums" formulae.* Take formulae for $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ as on page 2; subtract them instead of adding; hence find a formula for $\sin\alpha\sin\beta$. Starting with $\sin(\alpha+\beta)$, do something similar to find a formula for $\sin\alpha\cos\beta$,
6. *"Sums–to–products" formulae.* If $x = \alpha+\beta$ and $y = \alpha-\beta$, find α, β in terms of x, y. By substituting into $(*)$, find a formula for $\cos x + \cos y$. Use similar ideas to find formulae for $\cos x - \cos y$ and $\sin x + \sin y$.

ANSWERS.

1. (b) *Hint.* Start with $\cos 3\theta = \cos(2\theta + \theta)$.
2. $1 + \tan^2\theta = \sec^2\theta$.
4. We have
$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\dfrac{\sin\beta}{\cos\beta}} \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\ .\end{aligned}$$
5. The other two "products–to–sums" formulae are
$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}\ ,$$
$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}\ .$$
6. We obtain $\alpha = \dfrac{x+y}{2}$, $\beta = \dfrac{x-y}{2}$ and so
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\ ,$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\ ,$$
$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\ .$$