

## SOLVING TRIGONOMETRIC EQUATIONS

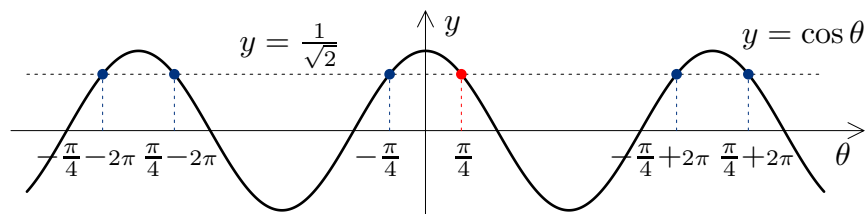
You will often need to be able to find all solutions of simple equations involving trigonometric functions. For example, solve

$$\cos \theta = \frac{1}{\sqrt{2}}.$$

First, you should know that one solution is  $\theta = \frac{\pi}{4}$ : if you are unsure about this, see the “values of trig functions” revision worksheet. However, this is not the only solution. We know that cosine is an even function, and so  $\theta = -\frac{\pi}{4}$  is also a solution. Also, cosine has period  $2\pi$ , so taking a solution and adding  $2\pi$  any number of times will give further solutions. So we have solutions

$$\theta = \pm \frac{\pi}{4} + 2n\pi \quad \text{where } n \text{ is an integer.}$$

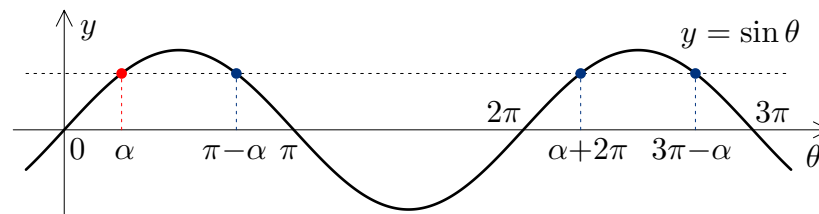
Looking at the graph may help you to find everything after the initial solution, and should also convince you that we have now found all possible solutions.



A similar procedure works for  $\cos \theta = -\frac{1}{\sqrt{2}}$ , though note that in this case our initial value will be in the second quadrant. If we want to solve  $\cos \theta = a$ , where  $a$  is not a “nice” value, then the solution will be written in terms of the inverse cosine function,

$$\theta = \pm \cos^{-1} a + 2n\pi \quad \text{where } n \text{ is an integer.}$$

Equations with sine instead of cosine work in much the same way. However sine is not an even function, so after finding an initial solution  $\theta = \alpha$ , the next solution is not  $-\alpha$ . By using the identity  $\sin(\pi - \theta) = \sin \theta$ , or by looking at the graph, you should

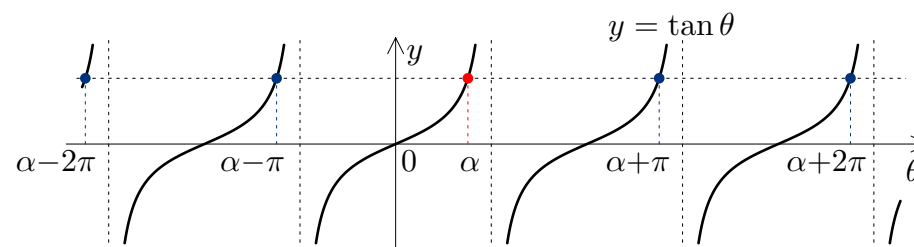


be able to see that the next solution is  $\pi - \alpha$ ; then for the complete solution, add multiples of  $2\pi$  as in the cosine case. For example, the complete solution of  $\sin \theta = \frac{1}{2}$  is

$$\theta = \frac{\pi}{6} + 2n\pi \text{ or } \theta = \frac{5\pi}{6} + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

**Note** that the equations  $\cos \theta = a$  and  $\sin \theta = a$  have no solution unless  $-1 \leq a \leq 1$ .

The graph of the tangent function does not oscillate like cosine and sine, but is always increasing (with “breaks”). So we’ll



have only one basic solution and will then apply periodicity: remember that the tangent function has period  $\pi$ , not  $2\pi$ . Thus, the solution of  $\tan \theta = a$  is

$$\theta = \tan^{-1} a + n\pi, \quad \text{where } n \text{ is an integer.}$$

## EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop-in Centre.

1. Solve the following equations. Give your answers in terms of exact values (fractions of  $\pi$ ) where possible.

(a)  $\cos \theta = \frac{1}{2}$ ;                      (b)  $\sin \theta = \frac{\sqrt{3}}{2}$ ;

(c)  $\cos \theta = -\frac{1}{\sqrt{2}}$ ;                      (d)  $\tan \theta = 4$ ;

(e)  $\sin \theta = 5$ ;                      (f)  $\cos \theta = \frac{2}{3}$ ;

(g)  $\tan \theta = \sqrt{3}$ ;                      (h)  $\tan \theta = -1$ ;

(i)  $\sin \theta = -\frac{1}{2}$ ;                      (j)  $\sin \theta = -\frac{1}{4}$ .

2. For question 1, parts (a)–(d), write down all solutions in the range  $0 \leq \theta < 2\pi$ .

3. Solve the equation  $\sin \theta = 1$ . How is this slightly different from the examples in the text?

4. Solve

(a)  $\sin(3\theta) = \frac{1}{2}$ ;                      (b)  $\tan(5\theta) = 1$ ;

(c)  $\cos(2\theta) = -\sqrt{3}$ ;                      (d)  $\cos(\frac{1}{3}\theta) = \frac{1}{3}$ .

5. Solve the following equations by rewriting them in terms of  $\tan \theta$ :

(a)  $2 \cos \theta - 3 \sin \theta = 0$ ;                      (b)  $4 \sin \theta + 7 \cos \theta = 0$ .

6. Solve the following by rewriting them in terms of  $\cos$ ,  $\sin$  or  $\tan$ :

(a)  $\sec \theta = \sqrt{2}$ ;                      (b)  $\cot(2\theta) = \sqrt{2}$ .

## ANSWERS.

In all the following,  $n$  is an integer: we have left this out in order to save space but you should **always** specify it in your answers.

- (a)  $\theta = \pm \frac{\pi}{3} + 2n\pi$ ;

(b)  $\theta = \frac{\pi}{3} + 2n\pi$  or  $\theta = \frac{2\pi}{3} + 2n\pi$ ;

(c)  $\theta = \pm \frac{3\pi}{4} + 2n\pi$ ;                      (d)  $\theta = \tan^{-1} 4 + n\pi$ ;

(e) no solution;                      (f)  $\theta = \pm \cos^{-1}(\frac{2}{3}) + 2n\pi$ ;

(g)  $\theta = \frac{\pi}{3} + n\pi$ ;                      (h)  $\theta = -\frac{\pi}{4} + n\pi$ ;

(i)  $\theta = -\frac{\pi}{6} + 2n\pi$  or  $\theta = \frac{7\pi}{6} + 2n\pi$ ;

(j)  $\theta = -\sin^{-1}(\frac{1}{4}) + 2n\pi$  or  $\theta = \sin^{-1}(\frac{1}{4}) + (2n+1)\pi$ .
- (a)  $\frac{\pi}{3}, \frac{5\pi}{3}$ ;                      (b)  $\frac{\pi}{3}, \frac{2\pi}{3}$ ;

(c)  $\frac{3\pi}{4}, \frac{5\pi}{4}$ ;                      (d)  $\tan^{-1} 4, \pi + \tan^{-1} 4$ .
- This is slightly different because the “first solution”  $\theta = \frac{\pi}{2}$  and the “second solution”  $\theta = \pi - \frac{\pi}{2}$  are actually the same. The full solution is  $\theta = \frac{\pi}{2} + 2n\pi$ .
- (a)  $\theta = \frac{\pi}{18} + \frac{2}{3}n\pi$  or  $\theta = \frac{5\pi}{18} + \frac{2}{3}n\pi$ ;

(b)  $\theta = \frac{\pi}{20} + \frac{1}{5}n\pi$ ;

(c) no solution;

(d)  $\theta = \pm 3 \cos^{-1}(\frac{1}{3}) + 6n\pi$ .
- (a)  $\tan \theta = \frac{2}{3}$ , so  $\theta = \tan^{-1}(\frac{2}{3}) + n\pi$ ;

(b)  $\tan \theta = -\frac{7}{4}$ , so  $\theta = -\tan^{-1}(\frac{7}{4}) + n\pi$ .
- (a)  $\cos \theta = \frac{1}{\sqrt{2}}$ , so  $\theta = \pm \frac{\pi}{4} + 2n\pi$ ;

(b)  $\tan(2\theta) = \frac{1}{\sqrt{2}}$ , so  $\theta = \frac{1}{2} \tan^{-1}(\frac{1}{\sqrt{2}}) + \frac{1}{2}n\pi$ .