The University of New South Wales

School of Mathematics and Statistics

Mathematics Drop–in Centre

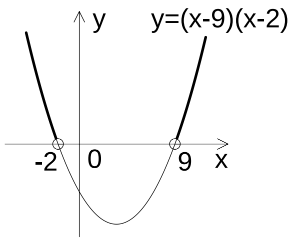
POLYNOMIAL INEQUALITIES

Polynomial inequalities such as

$$x^2 - 7x > 18$$

can be solved by collecting all terms on the left hand side to give $x^2 - 7x - 18 > 0$, and then factorising:

$$(x-9)(x+2) > 0 \ .$$



The graph of $y = (x-9)(x+2)$ is shown; we need to consider the case when $y > 0$, that is, the part of the graph above the x–axis, and find the corresponding x values. These x–values are given by

$$x < -2 \quad \text{or} \quad x > 9 \ , \qquad (*)$$

the solution of the inequality. **Note**. Do not write this solution as "$x < -2$, $x > 9$" or as "$9 < x < -2$". Each of these means "$x < -2$ and $x > 9$", which is not the same as $(*)$, and is **wrong**.

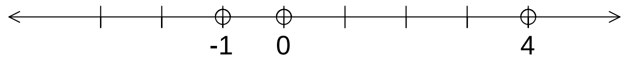
Inequalities where the unknown also appears in the denominator can be approached by multiplying out denominators *carefully* to obtain a polynomial inequality. For example, consider

$$x - 3 - \frac{4}{x} \le 0 \ .$$

First we must note that x cannot be 0, because of the third term. Now to clear the denominator we **do not** multiply by x: since x is unknown, we might be multiplying by a positive or a negative number, and we could not know whether we need to reverse the direction of the inequality. Instead, multiply both sides by x^2, which we know is positive. This gives $x^3 - 3x^2 - 4x \le 0$, and we can factorise the left hand side to obtain the inequality

$$x(x+1)(x-4) \le 0 \ .$$

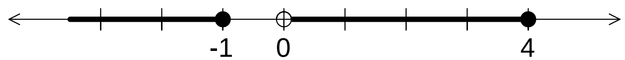
We can now graph $y = x(x+1)(x-4)$ and complete the solution as above. Alternatively, we may work with a number line, marking on it the points at which $y = 0$, that is, $x = -1, 0, 4$. These three



points divide the real line into four intervals and we consider each separately.

- If $x < -1$ then x and $x+1$ and $x-4$ are all negative; so their product is negative; so this interval is part of our solution.
- If $-1 < x < 0$ then $x+1$ is positive but x and $x-4$ are negative; so the product is positive; so this interval is not part of our solution.
- The intervals $0 < x < 4$ and $x > 4$ are treated in the same way.

Since the inequality is $\le$ rather than $<$, the endpoints $-1, 0, 4$ are included in the solution, *except* that we have already noted that $x = 0$ is impossible and must be excluded. Thus the solution can be illustrated as shown on the number line



and written algebraically as

$$x \le -1 \quad \text{or} \quad 0 < x \le 4 \ .$$

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Solve the following polynomial inequalities **both** by sketching a graph and by using a sign diagram.
   (a) $(x-3)(x-7) \geq 0$;
   (b) $(x-3)(x-7) < 0$;
   (c) $(x+4)(x-2)(x-9) > 0$;
   (d) $x^2 - 7x + 6 \leq 0$;
   (e) $x^2 + 4x < 12$;
   (f) $x^3 - 13x + 12 \leq 0$;
   (g) $x^3 - x^2 - 5x - 3 < 0$.
2. Solve the following by whichever method you find easiest.
   (a) $\dfrac{6}{x+1} \leq x+2$;
   (b) $2x + 5 - \dfrac{1}{x+3} < 0$;
   (c) $\dfrac{1}{x-2} \leq \dfrac{3}{x+4}$;
   (d) $\dfrac{x+2}{x-3} > 2$;
   (e) $x - 2 + \dfrac{x}{x-6} \geq 0$.

ANSWERS.

1. (a) $x \leq 3$ or $x \geq 7$;
   (b) $3 < x < 7$; can be written $x > 3$ and $x < 7$;
   (c) $-4 < x < 2$ or $x > 9$;
   (d) $1 \leq x \leq 6$;
   (e) $-6 < x < 2$;
   (f) $x \leq -4$ or $1 \leq x \leq 3$;
   (g) $x < -1$ or $-1 < x < 3$; alternatively, $x < 3$ and $x \neq -1$.
2. (a) $-4 \leq x < -1$ or $x \geq 1$;
   (b) $x < -\frac{7}{2}$ or $-3 < x < -2$;
   (c) $-4 < x < 2$ or $x \geq 5$;
   (d) $3 < x < 8$;
   (e) $3 \leq x \leq 4$ or $x > 6$.