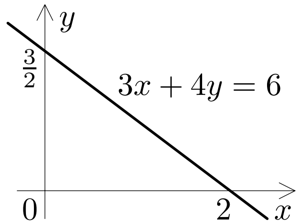
The University of New South Wales
School of Mathematics and Statistics

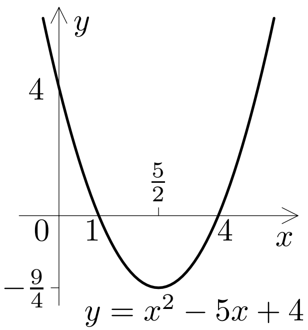
Mathematics Drop–in Centre

GRAPHS

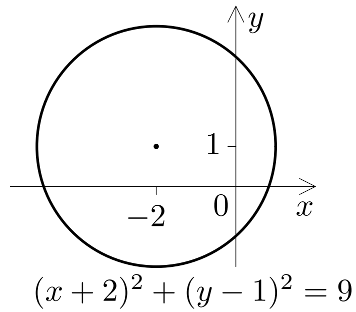
Graph sketching is a very important skill. From a well drawn graph you may be able to immediately see properties of a function, including roots, limits, turning points and where the function is increasing or decreasing. Graphs should always be **large** and **neatly drawn**, and important features should be **labelled**.

The equation $ax + by = c$ represents a **straight line**. Often the easiest way to sketch its graph will be to find its intercepts on the axes. For example, consider $3x + 4y = 6$. Substituting $x = 0$ and solving for y gives the y–intercept $(0, \frac{3}{2})$; letting $y = 0$ gives the x–intercept $(2, 0)$; we plot these points and draw the line joining them.





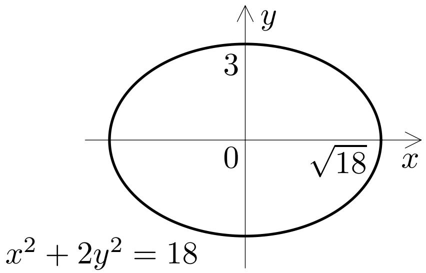
The quadratic equation $y = ax^2 + bx + c$ represents a **parabola**. To sketch the graph we need only find its roots (see revision worksheets on quadratics if you need) and note its concavity. For a more accurate sketch the y–intercept and vertex may also be useful. Consider, for example, $y = x^2 - 5x + 4$. There are x–intercepts at the roots of the quadratic, $x = 1$ and $x = 4$, and since x^2 has a positive coefficient the parabola is concave upwards. The y–intercept is $y = 4$. The vertex has x–coordinate halfway between the roots, that is, at $x = \frac{5}{2}$, and by substituting this into the quadratic we obtain the y–coordinate $y = -\frac{9}{4}$.



The graph of $x^2 + y^2 = r^2$ is the **circle** having centre at the origin and radius r; if the centre of the circle is at $x = a$, $y = b$ instead, then its equation is $(x - a)^2 + (y - b)^2 = r^2$. For an example of the latter,

$$(x + 2)^2 + (y - 1)^2 = 9$$

is the equation of a circle with centre $(-2, 1)$ and radius 3.



The equation of an **ellipse** is similar to that of a circle, but with different (positive) coefficients on the x^2 and y^2 terms. The standard form of the equation is

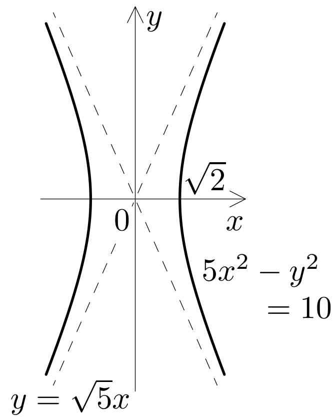
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 , \qquad (*)$$

for which the ellipse is centred at the origin, and has semi–axis lengths a in the x direction and b in the y direction. The best way to deal with an equation such as $x^2 + 2y^2 = 18$ is to divide both sides by 18 in order to put the equation into standard form, $(x^2/18) + (y^2/9) = 1$. This is an ellipse having semi–axis lengths $\sqrt{18}$ and 3.

If in the previous case the y^2 term has a negative coefficient, the curve is a **hyperbola**

$$(*) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

To sketch it, first draw the asymptotes given by $(x^2/a^2) - (y^2/b^2) = 0$, that is, $y = \pm bx/a$; then place the two branches of the curve. For example, $5x^2 - y^2 = 10$ has as asymptotes the lines $y = \pm\sqrt{5}x$; it has x–intercepts at $\pm\sqrt{2}$ and no y–intercepts.



EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Draw **large**, **neat**, **labelled** graphs of the following.

   (a) $5x + 4y = 32$; (b) $y = 2x^2 - 4x - 6$;

   (c) $x^2 + y^2 = 49$; (d) $y = -x^2 - 2x + 35$;

   (e) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$; (f) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$;

   (g) $2y = x + 6$; (h) $3x^2 + 8y^2 = 48$;

   (i) $(x-4)^2 + (y+2)^2 = 16$; (j) $x^2 - y^2 = 25$.

2. The equations $(*)$ can be modified to give ellipses and hyperbolas with centres away from the origin in much the same way as we did for circles. Find the centres of the following curves, and sketch them.

   (a) $\dfrac{(x-1)^2}{18} + \dfrac{(y+3)^2}{9} = 1$; (b) $5x^2 - (y-2)^2 = 10$.

3. What is the difference between the hyperbolas we have seen and one in which the coefficient of x^2, instead of y^2, is negative? Sketch

   (a) $-5x^2 + y^2 = 10$; (b) $4y^2 - x^2 = 16$.

4. The equation $xy =$ constant also defines a hyperbola, but it is positioned differently from those above. Sketch

   (a) $xy = 6$; (b) $xy = -10$.

   What are the asymptotes of these curves?

ANSWERS. In these answers we have *described* the graphs set in the exercises, but of course you need to actually *draw* them!

1. (a) A line with intercepts $(\frac{32}{5}, 0)$ and $(0, 8)$.

   (b) A parabola, concave upwards, x–intercepts at -1 and 3, vertex at $(1, -8)$ and y–intercept at -6.

   (c) A circle, centre at the origin, radius 7.

   (d) A parabola, concave downwards, x–intercepts at -7 and 5, vertex at $(-1, 36)$ and y–intercept at 35.

   (e) An ellipse, centre at the origin, semi–axis lengths 2 in the x direction and 5 in the y direction.

   (f) A hyperbola, centre at the origin, asymptotes $y = \pm\frac{4}{3}x$ and x–intercepts ± 3.

   (g) A line with x–intercept -6 and y–intercept 3.

   (h) An ellipse, centre at the origin, semi–axis lengths 4 in the x direction and $\sqrt{6}$ in the y direction.

   (i) A circle with centre $(4, -2)$ and radius 4. **Note** for your sketch: the circle will have the y axis as a tangent.

   (j) A hyperbola with centre at the origin; the asymptotes are $y = \pm x$ and the x–intercepts ± 5.

2. (a) An ellipse, as on page 2 but shifted to have centre $(1, -3)$.

   (b) A hyperbola, shape exactly as on page 2 but shifted to have centre $(0, 2)$. The asymptotes are $y - 2 = \pm\sqrt{5}\,x$.

3. The branches of the hyperbolas will be located at the "top and bottom" of the plane instead of "left and right".

4. The asymptotes will be the x and y axes. Curve (a) is a hyperbola in the first and third quadrants, (b) in the second and fourth. Turn your paper through 45° to see that they are the same shape as in the text and previous exercises.