The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

GEOMETRIC SERIES

A **geometric series** (or *geometric progression*, GP for short) is a sum of terms in which each and every number is a fixed ratio times the previous number. For example, in the sum

$$4 + 12 + 36 + 108 + 324 + 972 + 2916 \qquad (*)$$

every term is 3 times the previous term. However a series like

$$5 + 10 + 20 + 30 + 90$$

is not a geometric series because the second term is 2 times the first and the third is 2 times the second, but the fourth is $1\frac{1}{2}$ times the third (and the fifth is 3 times the fourth). A geometric series may have more, or fewer, terms than example $(*)$. The general geometric series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} ,$$

where every term is r times the previous one. We call a the **first term**, r the **common ratio** and n the **number of terms**. For instance, the geometric series $(*)$ has $a = 4$, $r = 3$ and $n = 7$. The sum of a geometric series can be found from the formula

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\,\frac{1 - r^n}{1 - r} ,$$

as long as $r \neq 1$. For example, we can work out $(*)$ as

$$4 + 12 + 36 + 108 + 324 + 972 + 2916 = 4\,\frac{1 - 3^7}{1 - 3} = 4372 .$$

You will often need to calculate the sum of a GP where the number of terms is unspecified; this can be done using exactly the same formula. For example, $4 + 4 \times 3 + 4 \times 3^2 + \cdots + 4 \times 3^{n-1}$ is a GP and its sum is

$$4\,\frac{1 - 3^n}{1 - 3} = 2(3^n - 1) .$$

Warning. You will need to take care with the number of terms. For example,

$$3 + 15 + 75 + 375 + \cdots + 3 \times 5^n$$

is a geometric series with $n + 1$ terms, not n, and its sum is

$$3\,\frac{1 - 5^{n+1}}{1 - 5} = \frac{3}{4}\left(5^{n+1} - 1\right) .$$

The ratio for a GP may be a fraction. For example

$$2 + \frac{2}{5} + \frac{2}{5^2} + \cdots + \frac{2}{5^{n-1}} = 2\,\frac{1 - \left(\frac{1}{5}\right)^n}{1 - \frac{1}{5}} = \frac{5}{2}\left(1 - \left(\frac{1}{5}\right)^n\right) .$$

It is also possible for a geometric progression to consist of an infinite number of terms. In this case it will still add up to a finite number as long as the ratio is less than 1 in absolute value. The formula in this case is

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1 - r} .$$

For example,

$$2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots = \frac{2}{1 - \frac{1}{5}} = \frac{5}{2} ;$$

however $4 + 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + \cdots$ does not add up to a finite value because the ratio $r = 3$ is bigger than 1.

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Which of the following are geometric series? If they are not, explain why not; if they are, write down the values of a, r and n and find the sum of the series.
   (a) $7 + 14 + 28 + 56 + 112 + 224 + 448 + 896$;
   (b) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45$;
   (c) $2 + 10 + 50 + 250 + 1250 + 12500$;
   (d) $3 + 3 \times 7 + 3 \times 7^2 + \cdots + 3 \times 7^{100}$.
2. Which of the following are geometric series? If they are not, explain why not; if they are, write down the values of a and r, and find the sum of the series if it is finite.
   (a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$;
   (b) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$;
   (c) $\frac{1}{6} + \frac{1}{30} + \frac{1}{150} + \frac{1}{750} + \cdots$;
   (d) $1 + 2 + 4 + 8 + 16 + 32 + \cdots$.
3. Find the sum of the following geometric series.
   (a) $7 + 7 \times 6 + 7 \times 6^2 + \cdots + 7 \times 6^{n-1}$;
   (b) $5 + 5 \times 11 + 5 \times 11^2 + \cdots + 5 \times 11^n$;
   (c) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$ (careful!).
4. The same formulae work if r is negative. Sum the following.
   (a) $2 - 6 + 18 - 54 + 162 - 486 + 1458$;
   (b) $3 - 3(\frac{2}{5}) + 3(\frac{2}{5})^2 + \cdots$;
   (c) $1 - 2 + 4 - 8 + 16 - 32 + \cdots$.

ANSWERS.

1. (a) $a = 7$, $r = 2$, $n = 8$, $S = 1785$;
   (b) not a GP since (e.g.) the first ratio is 3, the second is 2;
   (c) not a GP as the last ratio is 10 and all the others are 5;
   (d) $a = 3$, $r = 7$, $n = 101$, $S = \frac{1}{2}(7^{101} - 1)$.
2. (a) not a GP as the first ratio is $\frac{1}{2}$ and the second is $\frac{2}{3}$;
   (b) $a = 3$, $r = \frac{1}{3}$, $S = \frac{9}{2}$;
   (c) $a = \frac{1}{6}$, $r = \frac{1}{5}$, $S = \frac{5}{24}$;
   (d) $a = 1$, $r = 2$, sum is not finite.
3. (a) $\frac{7}{5}(6^n - 1)$;
   (b) $\frac{1}{2}(11^{n+1} - 1)$;
   (c) $8\left(1 - \left(\frac{1}{2}\right)^{n+3}\right)$.
4. (a) 1094;
   (b) $\frac{15}{7}$;
   (c) no finite sum because $r = -2$ and the absolute value of r is 2 which is greater than 1.