The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

ALGEBRAIC IDENTITIES

There are a number of algebraic identities which you need to know in order to help you solve equations and simplify expressions (by an *identity* we mean an equation involving one or more variables, which is true for all values of those variables).

- Addition of fractions:

$$\frac{w}{x} + \frac{y}{z} = \frac{wz + xy}{xz} \ .$$

- Square of a sum:

$$(x + y)^2 = x^2 + 2xy + y^2 \ .$$

- The same for a sum of three terms:

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \ .$$

- Difference of two squares:

$$x^2 - y^2 = (x - y)(x + y) \ .$$

- Difference of two powers:

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \ .$$

- Sum of two powers: if n is odd then

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}) \ .$$

  Note that there is *no* formula like this if n is even: for example, $x^2 + y^2$ cannot be factorised in any simple way.

- The Binomial Theorem to expand $(x+y)^n$: this is dealt with in a separate sheet.
- Power laws and logarithm laws such as $a^x a^y = a^{x+y}$: these are dealt with in separate sheets.

Comments.

- You must be able to use these identities "in both directions". For example, if you see $x^2 - y^2$ you should know that it can be factorised, and if you see $(x - y)(x + y)$ you should know that it can be expanded and simplified.
- You must be able to replace the variables in all these identities by different variables, constants or expressions. From the "difference of two squares" formula we can find, for example,

$$\begin{aligned} m^2 - n^2 &= (m - n)(m + n) \\ x^2 - 5 &= \big(x - \sqrt{5}\big)\big(x + \sqrt{5}\big) \\ 9a^2 - 100b^2 &= (3a - 10b)(3a + 10b) \\ x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \ . \end{aligned}$$

- Sometimes we can use a number of these identities successively in order to give a detailed factorisation of a certain expression. For instance,

$$\begin{aligned} x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \qquad \text{(difference of two squares)} \\ &= (x - y)(x^2 + xy + y^2)(x^3 + y^3) \\ &\qquad \text{(difference of 3rd powers)} \\ &= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2) \\ &\qquad \text{(sum of 3rd powers).} \end{aligned}$$

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Expand
   (a) $(4x+5y)(4x-5y)$; (b) $(s+t)^2$;
   (c) $(x-3y+5z)^2$; (d) $(z^3-4)(z^3+4)$;
   (e) $(ab-2cd)^2$; (f) $((a+b)-c)((a+b)+c)$;
   (g) $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$;
   (h) $(a+b+c+d)^2$ (i) $(x+y)^2-(x-y)^2$.
2. Factorise
   (a) $x^2+10xy+25y^2$; (b) x^6-y^4;
   (c) x^8-y^8; (d) $4a^2-5b^2$;
   (e) x^6-2x^3+1; (f) z^7-128 (hint: $128=2^7$).
3. (a) Factorise x^4-x^2+1 by writing it as $(x^4+2x^2+1)-3x^2$ and using the above identities.
   (b) Hence factorise $x^{12}-1$ into linear and quadratic factors.
4. The sum and difference of fractions

$$\frac{1}{x-1}-\frac{2}{x^2-1}+\frac{1}{x^3-1}$$

   has a common denominator $(x-1)(x^2-1)(x^3-1)$, but this is not the *smallest* common denominator. By factorising the three denominators, find the smallest common denominator and hence simplify the expression.

ANSWERS.

1. (a) $16x^2-25y^2$;
   (b) $s^2+2st+t^2$;
   (c) $x^2+9y^2+25z^2-6xy+10xz-30yz$;
   (d) z^6-16;
   (e) $a^2b^2-4abcd+4c^2d^2$;
   (f) $a^2+2ab+b^2-c^2$;
   (g) x^5+y^5;
   (h) $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$;
   (i) $4xy$.
2. (a) $(x+5y)^2$;
   (b) $(x^3-y^2)(x^3+y^2)$;
   (c) $(x^4-y^4)(x^4+y^4)$ for a start, but hopefully you can continue and get $(x-y)(x+y)(x^2+y^2)(x^4+y^4)$;
   (d) $(2a-\sqrt{5}b)(2a+\sqrt{5}b)$;
   (e) $(x^3-1)^2$; even better, $(x-1)^2(x^2+x+1)^2$;
   (f) $(z-2)(z^6+2z^5+4z^4+8z^3+16z^2+32z+64)$.
3. (a) $(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)$;
   (b) $(x-1)(x^2+x+1)(x+1)(x^2-x+1)(x^2+1)$
   $(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)$.
4. The smallest common denominator is $(x-1)(x+1)(x^2+x+1)$ and the expression is

$$\frac{x^3+x}{(x-1)(x+1)(x^2+x+1)}\,.$$