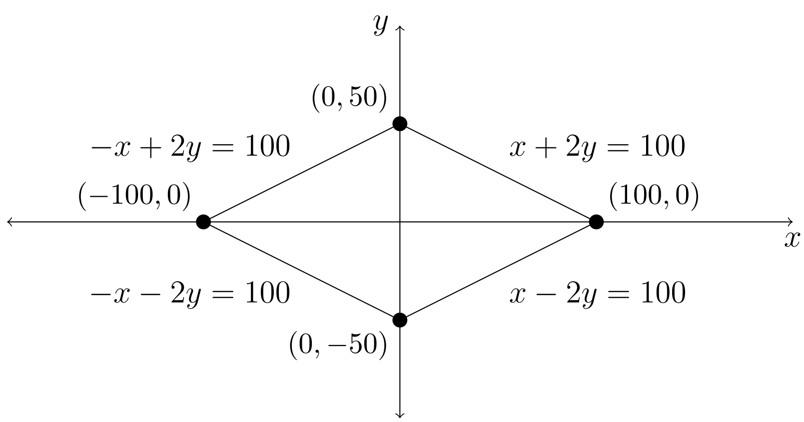
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 8, July 1, 2019

1. First note that x must be an even number, so let's consider the possible values of y.
If $y = 0$, then $x = \pm 100$ (2 solutions).
If $y = \pm 1$, then $x = \pm 98$ (4 solutions)
If $y = \pm 2$, then $x = \pm 96$ (4 solutions)
However, if $y = \pm 50$, then $x = 0$ (2 solutions again).
So there are $49 \times 4 + 2 \times 2 = 200$ solutions.

   The following graphical solution was contributed by a student.



   We can think of the solutions to the equation as being the integral points (points with integer coordinates) lying on the sides of the diamond in the number plane shown above. On any side of the diamond, there are 51 integral points. However, the four corner points lie at the end of two different sides. Thus there are $4 \times 51 - 4 = 200$ solutions.

2. Note that we can write the list of numbers as $2^0, 2^1, \ldots, 2^8$. In this new format, the product of any two number is 2 to the power of the sum of their exponents. Therefore, we can proceed to filling in the 3×3 product square as a slightly modified standard 3×3 magic square. To show whether the solution is unique, see solution sheet 1.

3. (a) x^3 must be an integer as $x^3 = 10 + 5[x]$. Also $x^3 \leq 10 + 5x$ and $x^3 > 10 + 5\{x - 1\}$ so $2 < x < 3$. Hence $[x] = 2$ and therefore $x^3 = 10 + 5 \times 2 = 20$ and $x = \sqrt[3]{20}$.

(b) $y^3 - 5\{y\} = 10$ so $10 < y^3 < 15$. Thus $2 < y < 3$, $y = 2 + \{y\}$ and so $\{y\} = y - 2$. Hence $y^3 = 10 + 5(y-2) = 5y$, and since $y \neq 0$, $y^2 = 5$ and hence $y = \sqrt{5}$.

4. There are two possibilities: either $f(x)$ is the product of a linear polynomial and a cubic or two quadratics. In the first case, this means that, for some integers a, b, c and d,

$$\begin{aligned} x^4 - nx + 63 &= (x+a)(x^3 + bx^2 + cx + d) \\ &= x^4 + (a+b)x^3 + (ab+c)x^2 + (ac+d)x + ad \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} a + b &= 0 && (1) \\ ab + c &= 0 && (2) \\ ac + d &= -n && (3) \\ ad &= 63 && (4) \end{aligned}$$

From (1), we have $b = -a$, which substituted into (2) gives $c = a^2$. If we substitute this into (3), we have $n = -(a^3 + d)$. Thus all the coefficients can be written in terms of a and d alone. Since $ad = 63$, both a and d have the same sign. We will consider them both negative, then the sign of n is positive and we can draw up the following table:

a	d	$n = -(a^3 + d)$
-1	-63	64
-3	-21	48
-7	-9	352
-9	-7	736
-21	-3	9264
-63	-1	250 048

In this case, the smallest value of n is 48.

Now let's consider the two quadratics case. Then

$$\begin{aligned} x^4 - nx + 63 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a+c)x^3 + (b+d+ac)x^2 + (bc+ad)x + bd \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} a + c &= 0 && (1) \\ b + d + ac &= 0 && (2) \\ bc + ad &= -n && (3) \\ bd &= 63 && (4) \end{aligned}$$

From (1), we have $a = -c$, which substituted into (2) gives $b + d = c^2$; and substituted into (3) gives

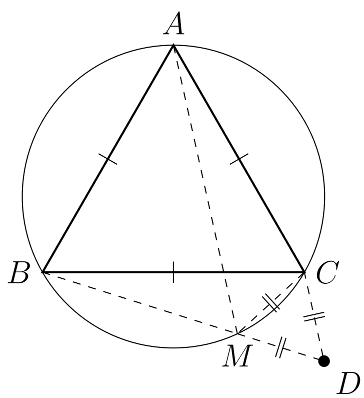
$$\begin{aligned} bc - cd &= -n \\ c(b-d) &= -n \\ n &= c(d-b) \end{aligned}$$

Thus we have

b	d	$c^2 = b + d$	c	$n = c(d-b)$
1	63	64	8	496
3	21	24	Not valid	
7	9	16	4	8

So the smallest value of n is 8.

5. Extend the line BM to the point D where $DM = CM$. Then $BD = MB + MC$. Since $ACMB$ is a cyclic quadrilateral and $\triangle ABC$ is equilateral, $\angle CMD = \angle BAC = 60°$. So $\triangle CMD$ is also equilateral. It can be shown by SAS that $\triangle ACM \equiv \triangle BCD$, and hence $AM = BD = MD + MC$.



Senior Questions

1. We have a solution to the equation when $x^2 - 5x + 5 = 1$, or when $x^2 - 5x + 5 = -1$ and $x^2 - 11x + 30$ is even, or when $x^2 - 5x + 5 \neq 0$ and $x^2 - 11x + 30 = 0$. For the first case, $x = 4$ and $x = 1$ are the only integral solutions. For the second case, $x = 2, 3$ are the solutions. For the last case, $x = 5, 6$ are solutions. There are six different solutions.

2. If $p_1, p_2, \ldots, p_n$ are n primes in arithmetic progression, then $p_1 = a, p_2 = a+d, \ldots, p_n = a + (n-1)d$. Also, if $p_1, \ldots, p_n$ are all greater than or equal to n, then d is divisible by every prime less than n. For if p is prime, $p < n$ and p does not divide d, then for some k with $1 \leq k \leq n$, $a + (k-1)d$ is divisible by p. But p_k is prime, so $p_k = p$. But $p < n, p_k \geq n$, a contradiction.

   So if $p_1, \ldots, p_6$ are all greater than or equal to 6, then d must be divisible by $2, 3$ and 5, so it is divisible by 30. So the smallest set of numbers we might consider is $7, 37, 67, 97, 127, 157$.

3. Since $2n+1$ is odd and a perfect square, we can write as $2n+1 = (2k+1)^2 = 4k^2+4k+1$, for some non-negative integer k, which implies $n = 2k(k+1)$. Since either k or $k+1$ is odd, we conclude that n is a multiple of 4.

   Because n is even, $3n+1$ must be odd so we can write $3n+1 = (2j+1)^2$, for some non-negative j, which implies $3n = 4j(j+1)$. Similar to before, either j or $j+1$ is odd, so we can conclude that n is divisible by 8.

   To complete the question, we show that n is divisible by 5. The possible remainder of an integer a divided by 5 are $0, 1, 2, 3$ and 4, therefore any perfect square number must have remainders $0, 1^2, 2^2, 3^2 - 5$ and $4^2 - 3(5)$; that is $0, 1, 4$ are the only remainders of a perfect square number when divided by 5. If we consider the remainders of $2n+1$ and $3n+1$ when divided by 5, for $n = 0, 1, 2, 3, 4$, we can see that the only time when both $2n+1$ and $3n+1$ have remainders either $0, 1$, or 4 is when $n = 0$. Hence the only time when both $2n+1$ and $3n+1$ are perfect squares is when $n \equiv 0 \pmod 5$; that is n is divisible by 5.