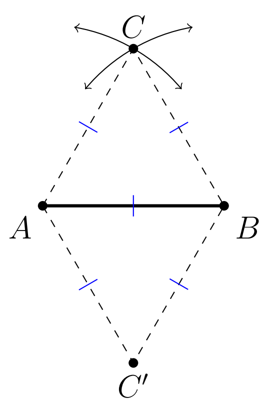
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 6, June 17, 2019[1]

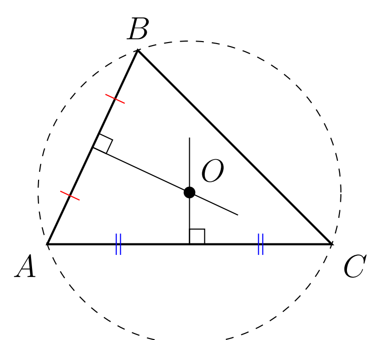
1. Some simple construction problems using straight-edge and compass techniques:

   (a) This one is pretty easy. Using the compasses, draw an arc centred at A with length AB. Draw a second arc of length AB but this time centred at B. The point where the two arcs cross is C, the apex of the equilateral triangle.



You will notice that there are two possible locations for C, one on either side of AB. The second one is shown as C' in the diagram. If you now join C and C', then you have drawn the perpendicular bisector of AB. This is because $ACBC'$ is a rhombus, and the diagonals of a rhombus bisect each other perpendicularly. We will use this method for 1(b).

   (b) The circumcentre (that is, the centre of the circumcircle) of any triangle is the point where the perpendicular bisectors of the three sides intersect. So, using the method given at the end of 1(a), we can construct the perpendicular bisectors of any two sides of the triangle. The point of intersection of the perpendicular bisectors is the circumcentre of $\triangle ABC$. Once we have found the circumcentre, we can easily draw the circumcircle.



[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

2. If we fix $x = 0$, then $y = 0, 1, 2, \ldots, 100$ so there are 101 choices for y. If we fix $x = 1$, then there are 100 choices for y, and so on. So the total number of ways to pick x and y such that $x + y \leq 100$ is equal to $1 + 2 + 3 + \ldots + 101 = 5151$.

3. It doesn't matter which prime you pick. If $p^2 + a^2 = b^2$ then

$$\begin{aligned} p^2 &= b^2 - a^2 \\ &= (b-a)(b+a). \end{aligned}$$

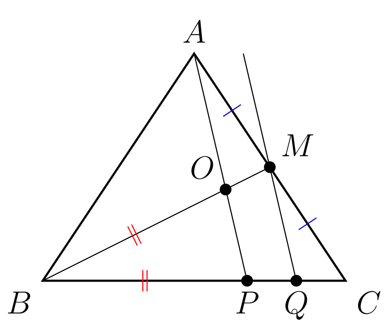
Because p is prime, the only divisors of p^2 is $1, p$ and p^2. Since a and b are integers, by the above equation, $b - a = 1$ and $b + a = p^2$, so that $\frac{a+b}{p} = p$.

4. Label the 21 people at the party by $a_1, a_2, \ldots, a_{21}$. Now a_1 knows at most four other people at the party, so by renumbering we can assume that a_1 does not know $a_6, a_7, \ldots a_{21}$. By renumbering again, we can assume that a_6 knows at most four of $a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}$, therefore a_1 and a_6 do not know $a_{11}, a_{12}, \ldots, a_{21}$. Similarly by renumbering, a_1, a_6 and a_{11} do not know $a_{16}, a_{17} \ldots, a_{21}$, and a_1, a_6, a_{11} and a_{16} do not know a_{21}. It follows that a_1, a_6, a_{11}, a_{16} and a_{21} do not know each other mutually.

5. Set $g(x) = f(x) - 2019$, then a_1, a_2, a_3, a_4, a_5 are the roots of $g(x)$, therefore we can write $g(x) = c(x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)h(x)$, where c is some constant and $h(x)$ a polynomial.

   Now the integral solutions to $f(x) = 2020$ are the integral solutions to $g(x) = 1$, but there is no integral solution to $g(x) = 1$, because in the expression $g(x) = c(x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)h(x)$, each $(x - a_i)$, $i = 1, 2, 3, 4, 5$ are distinct integers for any integer x. Also, $h(x)$ and c are integers for any integer x otherwise $f(x)$ will have non-integer coefficients; multiplying 7 integers in which at least 5 of are distinct can not give 1.

6. Draw a line parallel to AP that intersects the line BC at the point Q, as shown in the diagram below.



Note that the triangles $\triangle BOP$ and $\triangle BMQ$ are similar and isosceles, so $|OM| = |PQ|$. Now to find $\frac{|OM|}{|PC|}$, all we have to do is work out what portion of $|PC|$ is occupied by $|PQ|$.

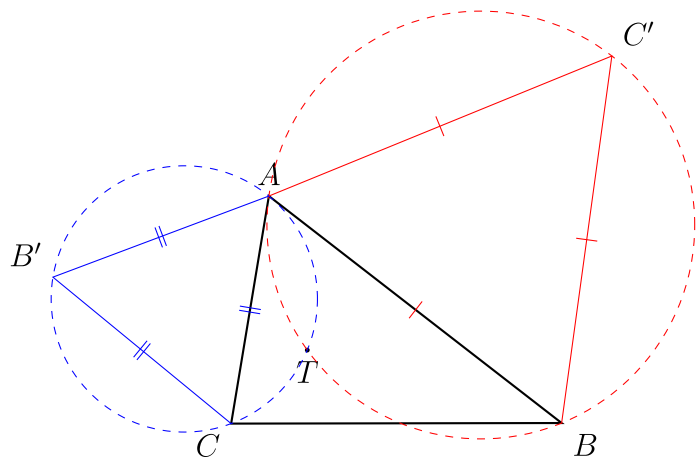
The triangles $\triangle ACP$ and $\triangle MCQ$ are similar, so we have $\frac{|AC|}{|PC|} = \frac{|MC|}{|QC|}$. But M is the midpoint of AC, which implies $|MC| = \frac{1}{2}|AC|$, so that

$$\frac{|AC|}{|PC|} = \frac{|MC|}{|QC|} = \frac{1}{2}\frac{|AC|}{|QC|}.$$

It follows that $2|QC| = |PC|$, which implies $2|PQ| = |PC|$, and therefore $\frac{|OM|}{|PC|} = \frac{1}{2}$.

Senior Questions

1. Pick any two sides of the triangle. In the diagram below, I have chosen AB and AC. Construct two equilateral triangles with these sides as a base. These are shown as $\triangle ABC'$ and $\triangle AB'C$ in the diagram. Find the two circumcircles of these triangles. (The method for both these steps is given in Q1 from the junior questions.)



One point of intersection of the two circles is the common vertex; the other is the point T, as can be shown using properties of cyclic quadrilaterals.

2. (a) $\pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$

   (b) If z is a fifth root of unity then

$$\begin{aligned} z^5 &= 1 \\ z^5 - 1 &= 0 \\ (z-1)(z^4+z^3+z^2+z+1) &= 0 \end{aligned}$$

   Since $z \neq 1$, $(z-1) \neq 0$, thus we may divide both sides by $(z-1)$ to obtain

$$z^4+z^3+z^2+z+1 = 0.$$

   (c) If $x = z + \frac{1}{z}$, then $x = z + z^{-1}$.

$$\begin{aligned} z &= \cos(\theta) + i\sin(\theta) \\ z^{-1} &= \cos(-\theta) + i\sin(-\theta) \qquad \text{(by De Moivre's theorem)} \\ &= \cos(\theta) - i\sin(\theta) \\ \therefore z + z^{-1} &= 2\cos\theta \end{aligned}$$

(d) As $z \neq 0$, we can divide (*) by z^2. Then

$$\begin{aligned}
z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} &= 0 \\
z^2 + 2 + \frac{1}{z^2} + z + \frac{1}{z} &= 1 \\
\left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) &= 1 \\
\therefore x^2 + x - 1 &= 0
\end{aligned}$$

(e) Applying the quadratic formula to $x^2 + x - 1 = 0$, we have $x = \frac{-1 \pm \sqrt{5}}{2}$. Since $\frac{2\pi}{5}$ is in the first quadrant, we take the positive solution, and thus $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$. Since $\frac{4\pi}{5}$ is in the second quadrant, we take the negative solution, and so $\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$.