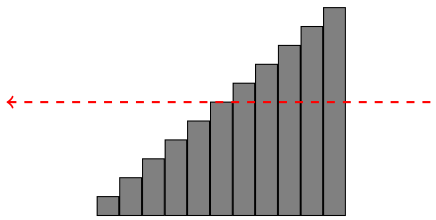
UNSW AUSTRALIA

Never Stand Still | Science | School of Mathematics and Statistics

MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 5, June 10, 2019[1]

1. You can think of an arithmetic sequence as a set of stairs, where the size of the"jump" from one step to the next is the same in every case.



As there is an odd number of terms in the arithmetic sequence, the middle term is equal to the average value of the sequence. That is

$$a_6 = \frac{S_{11}}{11} = \frac{220}{11} = 20.$$

2. (a) Let n be the total number of people at the party. Then the total number of handshakes is given by $(n-1)+(n-2)+(n-3)+\ldots+1$. This is the sum of an arithmetic sequence with $n-1$ terms. Thus

$$(n-1)+(n-2)+(n-3)+\ldots+1 = \frac{n-1}{2}[(n-1)+1] = \frac{n^2-n}{2}$$

If $\frac{n^2-n}{2} = 253$, then $n = 23, -22$. If we take the positive solution, then there are 23 people at the part in total (including Bernard), thus Bernard has invited 22 guests.

(b) Since there are 23 people in the party, we first divide them into groups of 12 and 11, the number of ways we can do this is

$$\frac{23!}{12! \times 11!},$$

where $23! = 23 \times 22 \times 21 \times \ldots \times 2 \times 1$, and similarly for $12!$ and $11!$. To see how the above equation works, $23!$ represent the number of ways we can put 23 people

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

into 23 chairs, but we don't care how the 12 or 11 people are arranged within the group, so we remove 12! and 11!.

Next we consider how many ways 11 and 12 people can be arranged at a round table. For the group of 12 people, there are 12! ways in which they can be arranged into 12 chairs. However because the table is round, we can rotate the table and get the same arrangement; the number of rotations is 12. Hence we conclude that there is $12!/12 = 11!$ ways the 12 person group can be arranged. For the group of 11 people, there are 10! ways to arrange them following a similar argument as the 12 group case. Thus we conclude the total number of ways Bernard can do this is

$$\frac{23!}{12! \times 11!} \times 11! \times 10! = \frac{23!}{12 \times 11}.$$

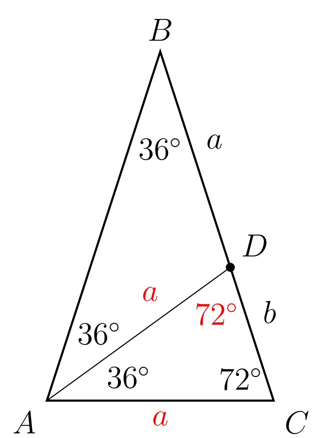
3. Suppose that n is a positive integer that is one less than a perfect cube. Then there is another positive integer m such that $n = m^3 - 1$. This is the difference of two cubes, thus

$$m^3 - 1 = (m-1)(m^2 + m + 1)$$

If $m = 2$, then this means that $n = 1 \times 7$, a prime. If, however, $m \neq 2$, then n has a non-trivial factorisation as neither $m - 1$ or $m^2 + m + 1$ is equal to one. That is, n is not prime.

4. Let $\triangle ABC$ be a isosceles triangle with base $\angle BAC = \angle BAC = 72°$, D is the point of intersection between the bisector of $\angle BAC$ and the line CB, as shown.

(a) Let $x = \frac{a}{b}$. By the sum of angles in $\triangle ADC$, $\angle ADC = 72°$. Therefore, the triangles $\triangle ACD$ and $\triangle BAC$ are similar. Furthermore, $\triangle ADB$ is isosceles, so that $a = AD = BD$, as shown below.



Now by ratios of similar triangles we have $\frac{AC}{CD} = \frac{BC}{AC}$ or $\frac{a}{b} = \frac{a+b}{a}$, which implies $x = 1 + \frac{1}{x}$. Solving for x gives $\frac{a}{b} = \frac{1+\sqrt{5}}{2}$.

(b) Applying the cosine rule to $\triangle ADC$, we have

$$\cos(36°) = \frac{2a^2 - b^2}{2a^2} = 1 - \frac{1}{2}\left(\frac{1}{x}\right)^2 = \frac{1+\sqrt{5}}{4}$$

5. We can always pair up the divisors of a number x in such a way that the product of the pair is equal to the number itself. For example the divisors of 4 are $1, 2$ and 4, which can be paired up into $\{1, 4\}$ and $\{2\}$. In particular, note that if x has an odd number of divisors, then x is a perfect square.

   Suppose we have a number x with an odd number of even divisors and even number of odd divisors, then the total number of divisors of x must be odd. Hence x is a perfect square.

   If x is odd, then is has no even divisors. Since zero is an even number, this means that x has an even number of even divisors, contrary to our assumption. Thus x is even.

   We can write x in terms of its prime factors as

$$x = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots p_k^{e_k}$$

   where the p_i for $i = 1, 2, \dots, k$ are the prime factors of x and the e_i are their respective exponents. In number theory, the τ-function is the function that returns the number of factors of a positive integer. Once the prime factorisation of x is known, $\tau(x)$ is easily computed.

$$\tau(x) = (e_1 + 1)(e_2 + 1)(e_3 + 1) \dots (e_k + 1).$$

   Now since x is even, we know that $p_1 = 2$, and to find the number of odd divisors, we should calculate $(e_2 + 1)(e_3 + 1) \dots (e_k + 1)$. However, as x is a square number, all the e_i are even numbers. This means that $(e_2 + 1)(e_3 + 1) \dots (e_k + 1)$, is the product of $k - 1$ odd numbers, and thus is odd. So there is no such number x.

6. Statement (c) is false. Because if we assume (c) is true, then by statement (b), $a + b = 3b + 5$ so that $3b + 5$ is divisible by 3; (b) is false. Also by statement (d), $7(a + b) - 6a$ is a prime, but $7(a + b) - 6a$ is divisible by 3; (d) is false. We have too many false statements if (c) is true.

   By statements (a) and (b), $2b + 6$ is divisible by b. Therefore b is a divisor of 6; that is b is $1, 2, 3$ or 6. By statement (b) and (d), $9b + 5$ is a prime. Then

b	$a = 2b + 5$	$a + 7b$
1	7	14
2	9	23
3	11	32
6	17	59

   We can see that the only prime numbers in the third column are 23 and 59, corresponding to the solutions $a = 9$, $b = 2$ and $a = 17$, $b = 6$.

Senior Questions

1. (a) One way to do this is by polynomial long division `http://en.wikipedia.org/wiki/Polynomial_long_division`, another way is by induction.

(b) Using part (a), we have $a^n - 1 = (a-1)(a^n + a^{n-1} + \ldots + a + 1)$. Since $a^n - 1$ is prime, the only factor it can have is 1; we must have $a - 1 = 1$, so $a = 2$.

Suppose n is not prime, then there are positive integers $x > 1$ and $y > 1$ such that $n = xy$. If we write $a^n - 1 = a^{xy} - 1 = (a^x)^y - 1$, then we can use the results of part (a) with a^x instead of a to obtain

$$a^n - 1 = (a^x)^y - 1 = (a^x - 1)[(a^x)^y + (a^x)^{y-1} + \ldots + a^x + 1].$$

Because the LHS of the above equation is a prime, we can conclude (just as before) that $a^x - 1 = 1$, which means $a^x = 2^x = 2$; $x = 1$, and we have a contradiction.

2. Let F_n be the nth Fibonacci number. The Fibonacci numbers, $0, 1, 1, 2, 3, 5, \ldots$, can be defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

(a) Substituting $F_n = Ar^n$ into the recurrence relation, we have

$$Ar^n = Ar^{n-1} + Ar^{n-2}.$$

Thus

$$\begin{aligned} Ar^n - Ar^{n-1} - Ar^{n-2} &= 0 \\ Ar^{n-2}(r^2 - r - 1) &= 0 \end{aligned}$$

Now if either $A = 0$ or $r = 0$, this would mean that $F_n = 0$ for all n, which is clearly not the case. Thus $r^2 - r - 1 = 0$, as required.

(b) As given in the question

$$F_n = \alpha \left(\frac{1+\sqrt{5}}{2} \right)^n + \beta \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Substituting in the initial conditions $F_0 = 0$ and $F_1 = 1$, we have

$$\alpha + \beta = 0 \qquad (1).$$

$$\alpha \left(\frac{1+\sqrt{5}}{2} \right) + \beta \left(\frac{1-\sqrt{5}}{2} \right) = 1 \qquad (2).$$

Solving this system of equations simultaneously gives the result $\alpha = \frac{1}{\sqrt{5}}$ and $\beta = -\frac{1}{\sqrt{5}}$. Thus the closed formula for the nth Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

(c) The first 5 Lucas numbers are

$$\begin{aligned} L_0 &= 2 \\ L_1 &= 1 \\ L_2 &= 3 \\ L_3 &= 4 \\ L_4 &= 7 \end{aligned}$$

You should obtain the formula

$$L_n = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n .$$