MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 4, June 3, 2019[1]

1. (a) Note that $8 = 2^3$ has a remainder of one when divided by 7. That is, we say that $2^3 \equiv 1 \bmod 7$. In modular arithmetic, if $a \equiv b \bmod n$ then $a^x \equiv b^x \bmod n$. Since $2^{2019} = 2^{3\times 673} = 8^{673}$, we can conclude that 2^{2019} has remainder 1 when it is divided by 7.

   (b) Let's look at the last digit of 2^n. That is, we will find $2^n \bmod 10$ for $n = 1, 2, 3, \ldots$.

$$\begin{aligned} 2^1 &\equiv 2 \bmod 10 \\ 2^2 &\equiv 4 \bmod 10 \\ 2^3 &\equiv 8 \bmod 10 \\ 2^4 &\equiv 6 \bmod 10 \\ 2^5 &\equiv 2 \bmod 10 \\ 2^6 &\equiv 4 \bmod 10 \\ &\vdots \end{aligned}$$

   We can see that this four-step pattern will keep repeating. Thus we can write

$$2^{2019} = 2^{4\times 504+3} \equiv 8 \bmod 10.$$

   Thus the last digit is 8.

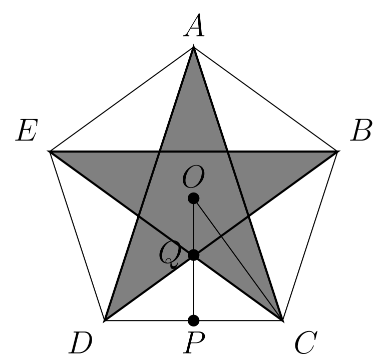
2. Gerald can roll either$\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 6\}$. The number of ways to obtain either of these combinations is $5 \times 4 \times 3 \times 2 \times 1 = 120$. The total number of outcomes with no restrictions is 6^5. Therefore the probability of obtaining 5 consecutive numbers is $\frac{2\times 120}{6^5} = \frac{240}{7776}$.

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

3.

$$\begin{aligned}
\sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+\dots}}}} &= 7\\
\sqrt{y+\sqrt{x+\sqrt{y+\dots}}} &= 7^2 - x\\
\sqrt{x+\sqrt{y+\dots}} &= (49-x)^2 - 2\\
7 &= (49-x)^2 - 2\\
x &= 46.
\end{aligned}$$

4. Let O be the centre of the pentagon, P the point of intersection between the bisector of the pentagon and one of it sides, and label the corners of the pentagon as shown below.



Since the pentagon is regular, we only need to look at the ratio of the shaded area of the $\triangle OPC$. Let Q be the point of intersection between CE and BD, then we are looking for the ratio of areas of $\triangle OPC$ and $\triangle OQC$. It is probably easier to find the area of $\triangle OQC$ by considering the difference of the areas of $\triangle OPC$ and $\triangle QPC$, as these two right angled triangles share a common base.

It can be shown that the internal angle of a regular pentagon is $108°$. Now, OC bisects $\angle DPC$, thus $\angle OCP = 54°$. Furthermore, $\triangle CDE$ is an isosceles triangle with $\angle CDE = 108°$. Thus $\angle PCQ = 36°$. Consequently, $OP = PC\tan 54°$ and $QP = PC\tan 36°$. Hence the ratio of the shaded region of $\triangle OPC$ is given by

$$\begin{aligned}
\frac{OP - QP}{OP} &= \frac{PC\tan 54° - PC\tan 36°}{PC\tan 54°}\\
&= 1 - \frac{\tan 36°}{\tan 54°}.
\end{aligned}$$

5. Let a and b be the two primes we get from adding up the numbers in the divided set. The only even prime number is 2, and the sum of any numbers in the list is greater than 2, it follows that a and b must be odd. There is only two odd numbers 39 and 45 from the list, so these must be put into different sets. Furthermore, if all numbers are divided by 3, their remainders are $0, 2, 0, 2, 0, 1, 0$ respectively. So to prevent a and

b from being a divided by 3, the numbers 38 and 44 must be in a different set to 46, thus we have two possibilities so far

$$A = \{38, 39, 44\}, B = \{45, 46\} \qquad \text{or} \qquad A = \{39, 46\}, B = \{38, 44, 45\}.$$

For the first case, the sum of A is 121 and the sum of B is 91, so we must add 24 to one set and 48 to the other. However, adding 24 or 48 to A does not give a prime number.

For the second case, the sum of A is 85 and B is 127. Since 85 is not prime, we must add 24, 48 or both to it. If we add 24 to 85, then we have to add the other number 48 to 127, which gives 175. But 175 is not a prime so this is not a solution. If we add 48 to 85, we get 133 which is not prime. Therefore the only solution is $\{24, 39, 46, 48\}$, $\{38, 44, 45\}$.

6. The last digit of x must be 0, because we need an integer after increasing x by 10%. Now in order to decrease the sum of digits of a number after increasing it by 10%, we look for a number that has a lot of digit that will "carry" up when multiplied by 1.1; an example would be a number starting with m lots of 3's, with n lots of 6's in the middle and a 0 at the end, because increasing this number by 10% gives a number with the $n-1$ lots of 3's, $m-1$ lots of 6's, one of each 7, 2 and 0. So the equation we need to solve is $(n-1)3 + (m-1)6 + 9 = 0.9(3m + 6n)$; $n = m = 10$ is a solution.

Senior Questions

1. The trick is to apply a change of variable, so that the two graphs become symmetrical. Let $X = x/10$ and $Y = 10y$, then $Y = 10\cos(10X)$ and $X = 10\cos(10Y)$. Let A be the sum of the new X-coordinate, and B the sum of the new Y-coordinate, then because the graph of X and Y are symmetrical, we have that $\frac{A}{B} = 1$. Now using the fact that the coordinates are positive, we have $A = a/10$ and $B = 10b$, therefore $\frac{a}{b} = 100$.

2. Apply a change of base to the logarithm.

3. First we consider case $0 < x \leq 1$. The RHS of $x = \frac{1}{2}\left(y + \frac{1}{y}\right)$ is the average of y and $1/y$, thus $y \leq x \leq \frac{1}{y}$ and $y \leq \frac{1}{x} \leq \frac{1}{y}$. Similarly, $z \leq y, \frac{1}{y} \leq \frac{1}{z}$, $t \leq z, \frac{1}{z} \leq 1/t$ and $x \leq t, \frac{1}{t} \leq 1/x$. From this we conclude that $x \leq t \leq z \leq y \leq x \leq \frac{1}{x} \leq \frac{1}{y} \leq \frac{1}{z} \leq \frac{1}{t} \leq \frac{1}{x}$, so the only solution for this case is $x = y = z = t = 1$.

   Using the same arguments as above, we can deduce that there is no solution for $1 < x$ and for the case $x < 0$, we have $x = y = z = t = -1$.

   To extend to the 2019 variable case, note that the above argument does not depend on the number of variables we had initially.