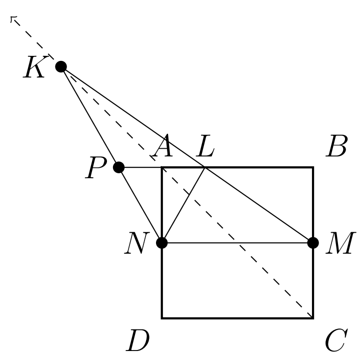
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 3, May 27, 2019

1. It is not very elegant, but the quickest way to solve this problem is probably brute force. That is, write out the first few powers of 2: $2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$. We notice that $2048 - 32 = 2016$. Consequently $a = 11$ and $b = 5$, so $a + b = 16$.

2. Let O be the midpoint of NM, extend the line AB so that it intercepts KN at the point P; see below. Since NM and PL are parallel and O is the mid point of NM, A is the midpoint of PL (this is a special case of the intercept theorem `http://en.wikipedia.org/wiki/Intercept_theorem`). Therefore the triangles PNA and ANL are congruent to each other, hence $\angle PNA = \angle ANL$.



3. We can write n as $n = 3^a 5^b 7^c \times N$, where the number N has no factors of 3, 5 or 7. Then $\frac{1}{3}n = 3^{a-1}5^b7^c \times N$, $\frac{1}{5}n = 3^a5^{b-1}7^c \times N$ and $\frac{1}{7}n = 3^a5^b7^{c-1} \times N$. Because we are looking minimal N, we may as well set $N = 1$. So for $\frac{1}{3}n$ to be a perfect cube, $\frac{1}{5}n$ to be a perfect fifth power and $\frac{1}{7}$ to be a perfect seventh power, we must have $a - 1$ a multiple of 3 and a itself a multiple of 5 and 7 (i.e., a multiple of 35). The smallest the smallest such a is 70. To find n, repeat this argument to obtain b and c.

4. We have
$$k^3 - 1 = (k-1)(k^2 + k + 1) = (k-1)(k(k+1) + 1)$$
and
$$k^3 + 1 = (k+1)(k^2 - k + 1) = (k+1)(k(k-1) + 1).$$
Therefore the numerator of the given product contains the factors $1, 2, 3, \ldots, n-1$ and the denominator contains $3, 4, 5, \ldots, n+1$. Most of these cancel and we are left with

$2/n(n+1)$. The numerator also contains factors $2\times 3+1, 3\times 4+1, \ldots, n(n+1)+1$,and the denominator $1\times 2+1, 2\times 3+1, \ldots, (n1)+1$; again most cancel and there remains $(n(n+1)+1)/(1\times 2+1)$. Combining all these results gives

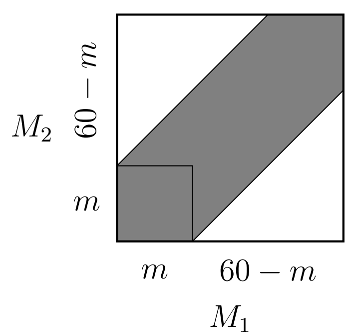
$$\frac{2^3-1}{2^3+1}\frac{3^3-1}{3^3+1}\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1}=\frac{2}{n(n+1)}\frac{n(n+1)+1}{1\times 2+1}=\frac{2}{3}\frac{n^2+n+1}{n^2+n}.$$

5. Let M_1 and M_2 be the two mathematicians. We can plot the arrival time of M_1 and M_2 on the $x-y$ plane, with x-axis representing the arrival time of M_1, and y-axis the arrival time of M_2; see figure **??**. Each mathematician stays in the tea room for exactly m minutes, so we know that if M_1 arrives first (say at 9 a.m.) then M_2 will run into M_1 in the cafeteria if M_2's arrival time is within m minutes of M_1; This is represented by the $m\times m$ square box in the bottom left of the plot. Over the break of 60 minutes, we get a shaded region as shown in figure **??**.

   The probability that either mathematician arrives while the other is in the cafeteria is 40%, thus the non-shaded region is 60% of the total area of the big square. So we have

$$\frac{(60-m)^2}{60^2}=0.6$$
$$m=60-12\sqrt{15},$$

   therefore, $a+b+c=87$.



6. Let $f(n)$ be the number of ways we can choose these n integers. We can try to workout what $f(n+1)$ is; that is the number of ways to choose $x_1, x_2, \ldots, x_n, x_{n+1}$ such that each is $0, 1$ or 2 and their sum even.

   Suppose we have n integers, $x_1, \ldots x_n$ from the list $0, 1, 2$ such that their sum is even. We know there is $f(n)$ ways to choose these n numbers, and we can either pick x_{n+1} to be 0 or 2 so that the sum of $x_1, \ldots, x_{n+1}$ is even; the total number of ways we can pick these $n+1$ integers is $2f(n)$.

   On the other hand, if the initial n integers, $x_1, \ldots x_n$ from the list $0, 1, 2$ is odd, then there is $3^n - f(n)$ ways to choose these n numbers, and we can only pick $x_{n+1}=1$ so that the sum of $x_1, \ldots x_{n+1}$ is even; the total number of ways we can pick these $n+1$ integers is $3^n - f(f)$

   Combining both cases, we have the recursive relation $f(n+1) = 3^n + f(n)$. Since it is straightforward to workout $f(1) = 2$, we can find $f(n)$.

Senior Questions

1. Given that a, b, and c are positive integers, solve

   (a) If $a > b$, then dividing both sides by $a!$, we have

   $$b! = \frac{b!}{a!} + 1,$$

   the LHS of the above equation is an integer, while the RHS is not; we have a contradiction on the condition $a > b$. We can apply the same arguments to get $a \not< b$, so that $a = b$. The only solution is then $a = b = 2$.

   (b) Notice this equation is symmetric in a and b, so we can assume without loss of generality $a \geq b$. Dividing through by $b!$, then

   $$a! = \frac{a!}{b!} + 1 + \frac{2^c}{b!}. \tag{1}$$

   The LHS of equation (1) is an integer and $a!/b!$ is an integer, therefore $2^c/b!$ must be an integer, this implies b is either 1 or 2. Also, the RHS of (1) is the sum of 3 integers, so $a!$ must contain a factor of 3; $a \geq 3$.
   If $b = 1$ then $a! = a! + 1 + 2^c$, which implies $2^c + 1 = 0$; there is no solution for c, so $b \neq 1$. Therefore $b = 2$.
   If $a > 3$, then $a!/2$ is even, so $2^{c-1} = 1$. But then we get $a!/2 = 2$, which has no solution for a.
   Therefore, we conclude that $a = 3$ and $b = 2$, therefore $c = 2$.

   (c)

2. (a) The inequality holds for $n = 3$. Assume $n! > (n-2)(1! + 2! + \ldots (n-)1!)$ and note that $2(n-2) \geq n-1$ for $n \geq 3$, therefore

   $$\begin{aligned}(n+1)! &= (n-1)n! + 2n! \\ &> (n-1)n! + 2(n-2)(1! + 2! + \ldots (n-1)!) \\ &\geq (n-1)(1! + 2! + \ldots + n!),\end{aligned}$$

   so the inequality holds for all n by standard induction arguments.

   (b) $(n+1)! < n(1! + 2! + \ldots + n!)$ because

   $$\begin{aligned}(n+1)! &= (n+1)n! \\ &= nn! + n! \\ &= n(n! + (n-1)!) \\ &< n(1! + 2! + \ldots + n!).\end{aligned}$$

   Therefore, combining with the result of (a),

   $$n < \frac{(n+1)!}{1! + 2! + \ldots + n!} < n + 1.$$

   So $(n+1)!$ divided by $1! + 2! + \ldots n!$ is a number that is strictly between n and $n+1$; $1! + 2! + \ldots n!$ does not divide $(n+1)!$.