MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 2, May 20, 2019

1. Let n be the number of vertices of the polygon. Then there are $n-3$ diagonals connected to each vertex, because the diagonals can not connect a vertex to itself or connect a vertex to either of the two vertices adjacent to it. Also, each diagonal connects exactly two vertices. Therefore, the total number of diagonals in a polygon is $n(n-3)/2$. To complete the question, solve the quadratic $n(n-3)/2 = 152$, which gives $n = 16$ or $n = -19$. Discard the unrealistic solution $n = -19$.

2. For $17p + 1$ to be a square, there must be some integer x such that $17p + 1 = x^2$. This implies

$$\begin{aligned} 17p + 1 &= x^2 \\ 17p &= x^2 - 1 \\ 17p &= (x-1)(x+1). \end{aligned}$$

   Consider the last line of this equation. Since both p and 17 are prime, their greatest common divisor is 1. This means the greatest common divisor between $(x-1)$ and $(x+1)$ is also 1. Therefore if $x - 1 = 17$ then $x + 1 = p$, and vice versa. Hence $p = 17 \pm 2$. The only solution is $p = 19$ because 15 is not prime.

3. Let $x(x+1)$ be the number of staff working for the politician, where x is some positive integer. Then the number of people making the school visits is $x(x+1) + 2 = x^2 + x + 2$. We will show that $x^2 + x + 2$ is not divisible by 3 using modular arithmetic in mod 3. Recall that doing arithmetic in mod 3 means that we are only interested in the remainder when an integer is divided by 3. For addition, subtraction and multiplication, this is the same as doing standard arithmetic except that we can reduce any large numbers to their remainder when divided by three.

   We only need to consider three cases, and let's put the results in the following table:

x	x^2	$x^2 + x + 2$
0	0	2
1	1	1
2	1	2

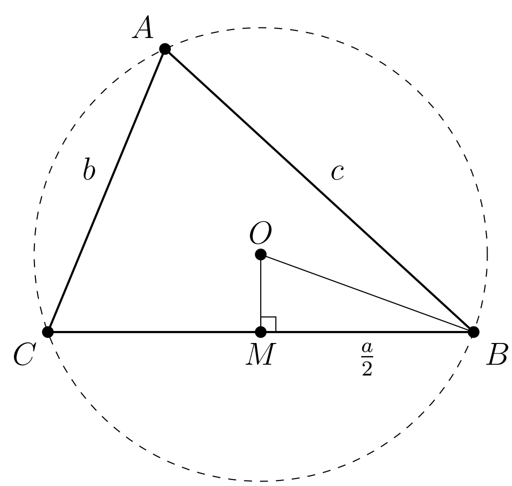
As the last column contains no zeros, we can see that there is no value of x for which $x^2 + x + 2$ is a multiple of three.

4. Let x, y and z be the total number of apples, peaches and mangoes respectively. If $a_1, a_2, \ldots, a_6$ are the number of apples in each basket, and $p_1, p_2, \ldots, p_6$ are the number of peaches in each basket. Then $p_1 = a_2 + a_3 + a_4 + a_5 + a_6$, $p_2 = a_1 + a_3 + +a_4 + a_5 + a_6$ etc. Therefore,

$$\begin{aligned} y =& p_1 + p_2 + \ldots + p_6 \\ &\begin{matrix} =0 & +a_2 & +a_3 & +a_4 & +a_5 & +a_6 & +\ldots \\ +a_1 & +0 & +a_3 & +a_4 & +a_5 & +a_6 & +\ldots \\ +\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & +\ldots \end{matrix} \\ =& 5(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) = 5x. \end{aligned}$$

We can repeat the above argument for apples and mangoes to get $x = 5z$. In conclusion, the total number of fruit is $x + y + z = x + 5x + 25x = 31x$.

5. (a) Let M be the midpoint of BC. Then OM is perpendicular to BC and bisects it. Thus $\triangle OMB$ is a right angled triangle with $MB = \frac{a}{2}$, $OB = \frac{d}{2}$.



Furthermore, as the angle at the centre is twice the angle at the circumference subtending the same chord, $\angle COB = 2A$ and thus $\angle MOB = A$. Consequently, $\sin A = \frac{a/2}{d/2}$ and the result follows.

(b) We can repeat a similar argument for the sides AC and AB to show that $\frac{b}{\sin B} = d$ and $\frac{c}{\sin B} = d$, respectively. As a result,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Senior Questions

1. If Alex cuts the cheese in the ratio $1 : x$, then this implies that the total amount of cheese is $1 + x$, and there are two chunks of cheese of size $\frac{1}{1+x}$ and $\frac{x}{1+x}$.

$\frac{1}{1+x}$	$\frac{x}{1+x}$

   After cutting the cheese twice, the pieces of cheese would be in the ratios $\frac{1}{1+x}$, $\frac{x}{(1+x)^2}$ and $\frac{x^2}{(1+x)^2}$.

$\frac{1}{1+x}$	$\frac{x}{(x+1)^2}$	$\frac{x^2}{(x+1)^2}$

   To divide the pieces of cheese such that there is an equal amount in each pile, we solve

$$\frac{1}{1+x} + \frac{x}{(x+1)^2} = \frac{x^2}{(x+1)^2},$$

   which has solutions $x = 1 \pm \sqrt{2}$. Since Alex can pick any number $x > 1$, he can divide the piles in to equal amounts in just two cuts by taking $x = 1 + \sqrt{2}$.

2. Let $f(x) = ax^n$, then $f'(x) = anx^{(n-1)}$. Remember that the inverse of f is defined as the function f^{-1} such that

$$f^{-1}(f(x)) = x, \text{ and } f(f^{-1}(x)) = x.$$

   Thus if the inverse and the derivative are equal, then

$$f'(f(x)) = a^2nx^{n(n-1)} = x.$$

   That is, $n(n-1) = 1$ and $a^2n = 1$. Solving these simultaneously, we obtain $n = (1 \pm \sqrt{5})/2$ and $a = \sqrt{2/(1 \pm \sqrt{5})}$. If we are looking for a real-valued function, then we must have $n = (1 + \sqrt{5})/2$ and $a = \sqrt{2/(1 + \sqrt{5})}$.

3. (a) Let F_n be the nth term of the sequence, then $F_{n+2} = F_{n+1} + F_n$; the next 3 terms of the sequence are $21, 34, 55$.

   (b)

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n \\
&= F_n + F_{n-1} + F_{n-1} + F_{n-2} \\
&= F_n + F_{n-1} + F_{n-2} + F_{n-2} + F_{n-3} \\
&= \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
&= F_n + F_{n-1} + \ldots + F_2 + F_2 + F_1 \\
&= S_n + F_2 = S_n + 1
\end{aligned}$$