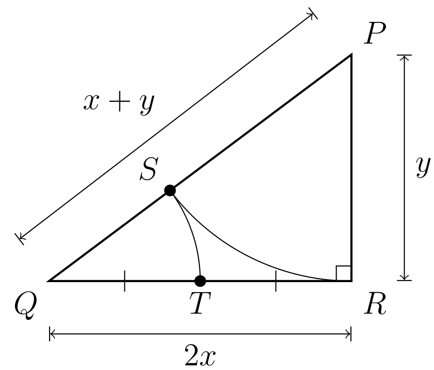
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 15 Solutions, September 17, 2019

1. Let $QS = x$ and $SP = y$. We want to find the value of x/y.



We can see that $PQ = x + y$, $QR = 2x$ and $PR = y$, so by Pythagoras' theorem,

$$\begin{aligned} y^2 + (2x)^2 &= (x+y)^2 \\ y^2 + 4x^2 &= x^2 + 2xy + y^2 \\ 3x^2 &= 2xy \\ \therefore \frac{x}{y} &= \frac{2}{3} \end{aligned}$$

So $QS : SP = 2 : 3$.

2. Since $x, y \neq 0$, we may multiply both sides of the equation by $4xy$ to clear out all the denominators. Then

$$\begin{aligned} 4x^2 + 4x + 4y &= xy \\ 4x^2 + 4x &= y(x-4) \\ \therefore y &= \frac{4x^2+4x}{x-4} \end{aligned}$$

By polynomial long-division (or otherwise), we can show that

$$\frac{4x^2+4x}{x-4} = 4x + 20 + \frac{80}{x-4}$$

Since y is an integer, this implies that $x-4$ is a factor of 80. As both x and y are positive, this means that the possible solutions are $(5,120)$, $(6,84)$, $(8,72)$, $(12,78)$, $(14,84)$, $(24,120)$, $(44,198)$, and $(84,357)$.

3. If the number is made from $a \neq 1$, then $aaa\ldots$ is divisible by a and thus not prime.

   Suppose the number has a non-prime number of digits, then we can factor the number of digit this number has into $q \times p$ where q and p are integers. Hence, we we can "split" the number up into q blocks of p-length digits; i.e

$$\underbrace{111\ldots1}_{p\times q \text{ lots of 1's}} = \underbrace{\underbrace{111\ldots1}_{p \text{ lots of 1's}} \times \underbrace{111\ldots1}_{p \text{ lots of 1's}} \times\ldots\times \underbrace{111\ldots1}_{p \text{ lots of 1's}}}_{q \text{ lots of blocks}}.$$

   The RHS of the number above is divisible by $\underbrace{111\ldots1}_{p \text{ lots of 1's}}$.

4. For a number to be divisible by 9, the sum of its digits must also be divisible by 9. If each digit of the number is even then so is the sum of its digits. So we start with smallest sum of digits that is divisible by 9 and even; this number is 18. It is easy to check that the number has at least 3-digits, so 288 is the smallest possible solution.

5. If we write $p(x)$ in the standard form, then

$$p(x) = a_{2020}x^{2020} + a_{2019}x^{2019} + \ldots + a_1x + a_0$$

   for some coefficients, $a_{2020}, \ldots, a_0$. To find the sum of these coefficients, we evaluate the polynomial at $x = 1$. Thus

$$\begin{aligned}
p(1) &= (1-2)^{2019}(1+2019) + (1-2)^{2018}(x+2018) + \ldots + (1-2)(1+1) \\
&= -2020 + 2019 - 2018 + 2017 - 2016 + \ldots + 5 - 4 + 3 - 2 \\
&= -2020 + (2019-2018) + (2017-2016) + \ldots + (5-4) + (3-2) \\
&= -2020 + 1009 \times 1 \\
&= -1011
\end{aligned}$$

6. Since the RHS of $(m-8)(m-10) = 2^n$ is a power of 2, both $m-8$ and $m-10$ must be powers of 2. Also, the difference between $m-8$ and $m-10$ is 2, so the two solutions are $m = 12$, $n = 3$ and $m = 6$, $n = 3$.

Senior Questions

1.
$$T_n\left(\frac{1}{T_1}+\frac{1}{T_2}+\ldots+\frac{1}{T_n}\right)=n^2$$

or

$$\frac{1}{T_1}+\frac{1}{T_2}+\ldots+\frac{1}{T_n}=\frac{2n}{n+1}.$$

This can be proven using induction. The inductive step depends on

$$\frac{2n}{n+1}+\frac{1}{(n+1)(n+2)/2}=\frac{2(n+1)}{(n+1)+1}.$$

2. Firstly, let's find the equation of the chord AB. Since this line passes through $A(a,a^2)$ and $B(b,b^2)$, the gradient is given by

$$m_{AB}=\frac{a^2-b^2}{a-b}=a+b.$$

Using the point gradient form of a line,

$$\begin{aligned} y-y_0 &= m(x-x_0) \\ y-a^2 &= (a+b)(x-a) \\ y &= (a+b)x-ab. \end{aligned}$$

Thus the y intercept of AB is $-ab$.

By a similar argument, we can show that the chord through CD has the equation $y=(c+d)x-cd$. As these two chords intersect on the y axis, $ab=cd$, and thus $d=\frac{ab}{c}$.