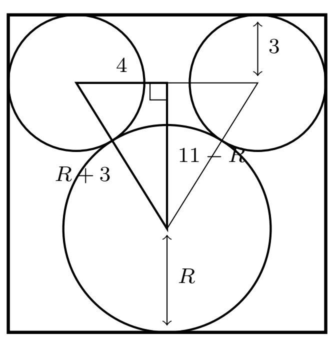
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 11 Solutions, August 20, 2019

1. Firstly note that

$$\begin{aligned}\frac{n^2+11n+2}{n+5} &= \frac{n^2+11n+30-28}{n+5}\\ &= \frac{(n+5)(n+6)}{n+5} - \frac{28}{n+5}\\ &= (n+6) - \frac{28}{n+5}\end{aligned}$$

This means that the LHS is an integer if $(n+5)$ is a factor of 28. The positive factors of 28 are $1, 2, 4, 7, 14, 28$, so the positive solutions for n are $2, 9, 23$.

2. Let R be the radius of the big circle. Draw a triangle that connects the centre of each circles, then bisect the this triangle into two right-angled triangles; as shown in the diagram.



From the diagram, we can see that the hypotenuse of the right-angled triangle has length $R+3$, and short sides of length 4 and $14-R-3 = 11-R$. Now by Pythagoras, we have

$$4^2 + (11-R)^2 = (R+3)^2.$$

Solving this equation gives $R = \frac{32}{7}$.

3. The prime number $p = 5$ works, because 101 and 151 are both prime numbers. We show that this is the only prime that works. First note that the numbers with remainders

$1, 2, 3$ or 4 when divided by 5 covers all possible integers not divisible by 5; i.e suppose the number x_1 has remainder 1 when divided by 5, then x_1 must be one of $1, 6, 11, \ldots$.

Now to check that $4x_1^2 + 1$ and $6x_1^2 + 1$ are not prime, we argue has follows: since x_1 has remainder 1 when divided by 5, $4x_1^2 + 1$ has remainder $4 \times 1^2 + 1 = 5$ when divided by 5. Hence $4x_1^2 + 1$ is divisible by 5 and therefore not prime. We can repeat this for the other numbers x_2 (which has remainder 2 divided by 5), x_3 and x_4, and show that it fails to be the desired prime number each time.

4. Arrange the stones clockwise in a circle ordered by increasing weight. Now move the stickers on each stone one position counter-clockwise. The result is that the heaviest stone is now labelled with the lightest weight, with all the other stones labelled with a sticker heavier than its true weight.

   Now if we are to select any number of stones from $1 - 99$, as long as we exclude the heaviest stone from our selection, the total true weight of the stones we selected will always be lighter than the total weight on their stickers. On the other hand, the set of stones we did not select (that is, the complementary subset containing the heaviest stone) will always have true weight less than the total weight given by their stickers, because the total weight of the stones has not changed.

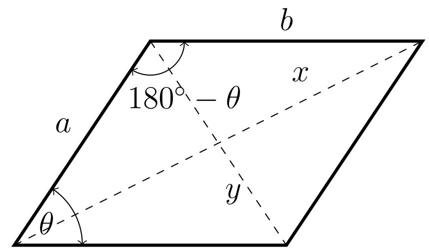
5. The 99 locked boxes can all be opened as long as the key to the smashed box is contained in the very last box we open.

   (a) We can think about this as placing 99 balls in a bag, with one of the ball painted red. The red ball represents the box that contains the key to the smashed box (we consider 99 boxes here because the first ball we draw is the box we smash). Hence, we want to draw the red ball last; the probability that this occurs is $1/99$.

   (b) We now have two red balls in the bag, so the probability of opening all locked boxes is $2/99$.

Senior Questions

1. (a) Let one of the internal angles of the parallelogram be θ. Then the other internal angle is $180° - \theta$.



By the cosine rule,

$$y^2 = a^2 + b^2 - 2ab\cos(\theta). \qquad (1)$$

Similarly,

$$x^2 = a^2 + b^2 - 2ab\cos(180° - \theta),$$

but since $\cos(180° - \theta) = -\cos(\theta)$,

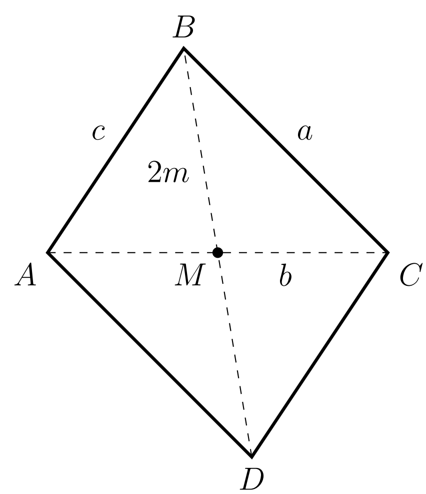
$$x^2 = a^2 + b^2 + 2ab\cos(\theta). \qquad (2)$$

Adding equations (1) and (2) together, we obtain

$$x^2 + y^2 = 2a^2 + 2b^2,$$

as required.

(b) Find the point D so that $ABCD$ is a parallelogram, as shown.



Then the two diagonals AC and BD intersect at M and $BD = 2m$. Applying the parallelogram law from part (a) to $ABCD$, we have

$$\begin{aligned}(2m)^2 + b^2 &= 2a^2 + 2c^2 \\ 4m^2 &= 2a^2 + 2c^2 - b^2\end{aligned}$$

Thus $m = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$. Furthermore, if $a = b = c = s$, then

$$m = \frac{1}{2}\sqrt{2s^2 + 2s^2 - s^2} = \frac{\sqrt{3}}{2}s.$$

2. The example shows that -4 is in T. We have further that -1 is in T, because $-1 = -(5-4) = -(0+1)$. Also -3 is in T, because $-3 = -(4-1) = -(0+3)$. Continuing in this way, we can eventually obtain $\{-5, -4, \ldots, 4, 5\} \in T$; that is the integers from -5 to 5 are all elements of the set T.

   Now to show that every integer is in T, we use proof by induction as follows. Suppose the set of integers $\{-n, \ldots, n\}$ is in T for $n \geq 5$. We have already shown that the basis case $n = 5$ is true. It remains to show that $-n-1$ and $n+1$ is also in T. Since $n+1 = -[(-1)+(-n)] = -[(-2)+(-n+1)]$, $n+1$ is in T. It follows that $-n-1$ is also in T, because $-n-1 = -[n+1+0] = -[(n-1)+2]$. Thus the result is proved by the principle of mathematical induction.