Parabola Volume 55, Issue 3 (2019)

58th Annual UNSW School Mathematics Competition: Competition Problems and Solutions

Solutions by Denis Potapov[1]

A Junior Division – Problems

Problem A1:
A rectangular piece of paper is cut along a straight line providing two polygons. One of the polygons is cut along a straight line, providing more polygons. Again, one of the polygons is cut along a straight line, providing more polygons, and this is repeated. Prove that if this process is repeated a sufficient number of times, then one will obtain 100 polygons with identical numbers of vertices.

Problem A2:[2]
A positive integer is called *perfect* if it equals the sum of all its proper divisors. For example, 6 and 28 are perfect numbers as

$$6 = 1 + 2 + 3 \quad \text{and} \quad 28 = 1 + 2 + 4 + 7 + 14\,.$$

It is not known whether or not there are any odd perfect integers. Prove that if an odd perfect integer does exist, then it cannot have each of 3, 5 and 7 among its divisors.

Problem A3:[3]
In an even-money game, a gambler is returned their wager plus an equal amount if they win, and they forfeit their wager if they lose.

A gambler cashes \$2500 in chips at a casino and plays an even-money game, making an initial wager of \$1. He then follows a strategy of tripling the wager on his next bet following a loss, and waging \$1 on their next bet following a win. All winnings, including wagers, are placed in the bag. At the start of the game the bag is empty.

For instance, if he had bet \$1 and won, then \$2 was placed in the bag. If he had lost the \$1 bet, then he would bet \$3 next. If the bet had been won, then he would have placed \$6 into the bag. If the \$3 bet had been lost, he would have bet \$9 next, and so on.

He stopped when he won exactly 100 bets and used up all the \$2500 chips at the same time. The gambler lost his bag but fortunately it was handed in to the lost property department. To claim the bag, the gambler had to state exactly how much was in the bag. The gambler consulted a mathematician friend. She correctly calculated the amount and the bag was returned.

How much was in the bag?

[1]Denis Potapov is a Senior Lecturer in the School of Mathematics and Statistics at UNSW Sydney.
[2]This problem was suggested by Dimitry Zanin.
[3]This problem was suggested by Sin Keong Tong.

Problem A4:
A square board tiled with 9×9 squares was filled with positive integers, one in each square, in such a way that: (a) the difference between every two numbers in squares symmetric with respect to the main diagonal is a multiple of 9; and (b) no difference between two numbers in the same row is a multiple of 9.

Prove that the sum of the numbers on the main diagonal is a multiple of 9.

Problem A5:[4]
At a theme park there is a game where contestants throw a circular disk with diameter 3.8cm onto a rectangular floor completely tiled with 4cm by 4cm squares. A prize is awarded if the disc lands completely inside a tile. Assuming that the division lines between tiles are infinitely thin, what is the probability of success?

Problem A6:
A bug sits at a corner of a rectangular box. The dimensions of the box are $1 \times 1 \times 2$. The bug can only travel on the faces and edges of the box. Find the shortest path for the bug to travel from its corner to the diagonally opposed corner.

B Senior Division – Problems

Problem B1:
Two players, Player A and Player B, play the following game. A set of 20 points is marked on a circle. The players take turns to connect two points on the circle with straight line segments, and with the additional constraint that the straight line segment is not permitted to intersect any prior straight line segment. Player A goes first.

Who should win and what strategy should they adopt?

Problem B2:
You are given ten bags containing hundreds of coins. You are reliably informed that nine of the bags contain genuine coins, each weighing 10g, and the remaining bag contains counterfeit coins, each weighing 11g. You are provided with a scale with a display and you can remove coins from the bags to weigh them, either individually or collectively, but you are only permitted to make one weight measurement. How can you identify the bag with counterfeit coins from the one weight measurement?

[4]This problem was suggested by Sin Keong Tong.

Problem B3:[5]
In an even-money game, a gambler is returned their wager plus an equal amount if they win, and they forfeit their wager if they lose.

A gambler cashes \$2500 in chips at a casino and plays an even-money game, making an initial wager of \$1. He then follows a strategy of tripling the wager on his next bet following a loss, and waging \$1 on their next bet following a win. All winnings, including wagers, are placed in the bag. At the start of the game the bag is empty.

For instance, if he had bet \$1 and won, then \$2 was placed in the bag. If he had lost the \$1 bet, then he would bet \$3 next. If the bet had been won, then he would have placed \$6 into the bag. If the \$3 bet had been lost, he would have bet \$9 next, and so on.

He stopped when he won exactly 100 bets and used up all the \$2500 chips at the same time. The gambler lost his bag but fortunately it was handed in to the lost property department. To claim the bag, the gambler had to state exactly how much was in the bag. The gambler consulted a mathematician friend. She correctly calculated the amount and the bag was returned.

How much was in the bag?

Problem B4:[6]
Let $\triangle ABC$ be a triangle and let every angle of $\triangle ABC$ be acute.

Prove that
$$R \leq \frac{AB + BC + CA}{4},$$
where R is the radius of the circumscribed circle.

Problem B5:
A tie-break game is played between two tennis players. The players take alternate serves and the tie-break game is won when one of the players wins on two successive serves. Each player has the same probability p of winning on their own serve. One of the players has just won a serve and it is their turn to serve next.

Find the probability that the other player wins the tie-break game.

[5] This problem was suggested by Sin Keong Tong.
[6] This problem modelled on a suggestion by Dimitry Zanin.

A Junior Division – Solutions

Solution A1.

If you cut a polygon along a straight line, then the two new polygons will have the same or fewer number of vertices except if one of the resulting parts is a triangle.

Hence, one can either find 100 triangles, or there are at most 99 triangles, in which case only the largest number of vertices observed on a polygon is $4 + 99 = 103$.

In the latter case, if the process is continued for sufficiently many steps, one will observe two polygons with identical vertices. □

Solution A2.

Assume the contrary. That is, assume that $n \in \mathbb{N}$ is such that n is odd and

$$n = 3^k 5^l 7^m a, \quad k \geq 1, l \geq 1, m \geq 1$$

and $a \in \mathbb{N}$ satisfies

$$\gcd(a, 3) = \gcd(a, 5) = \gcd(a, 7) = 1 \, .$$

Note that if $m, m_1, m_2 \in \mathbb{N}$ and

$$m = m_1 m_2 \quad \text{and} \quad \gcd(m_1, m_2) = 1 \, ,$$

then

$$S(m) = S(m_1) \times S(m_2) \, ,$$

where $S(m)$ is the sum of all divisors of m.

Let us show first that $m \geq 2$ and $k \geq 2$. Indeed, if $k = 1$, then $S(3) = 4$, and

$$4 / S(n) = 2n \, .$$

This contradicts the assumption that n is odd.

Similarly, if $m = 1$, then $S(7) = 8$, and

$$8 / S(n) = 2n \, .$$

This again contradicts to the assumption that n is odd.

From now on, we assume that $m \geq 2$ and $k \geq 2$. We then have that

$$\begin{aligned}
2n = S(n) &= S(3^k) \cdot S(5^l) \cdot S(7^m) \cdot S(a) \\
&\geq \left(3^k + 3^{k-1} + 3^{k-2}\right) \times \left(7^m + 7^{m-1} + 7^{m-2}\right) \times \left(5^l - 5^{l-1}\right) \times a \\
&\geq n \times \left(1 + \frac{1}{3} + \frac{1}{9}\right) \times \left(1 + \frac{1}{7} + \frac{1}{49}\right) \times \left(1 + \frac{1}{5}\right) \\
&> 2n \, .
\end{aligned}$$

Therefore, the initial assumption that n is odd and has 3, 5 and 7 among its divisors is false. □

Solution A3.
Note that if the gambler starts with bet \$1, experiences $n-1$ losses and wins on n-th turn, then the amount of money used is

$$M = 1 + 3 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}\,.$$

At the end of this cycle, the gambler adds to the bag $W = 2 \times 3^{n-1}$ dollars. That is,

$$W = 2 \times \frac{2M + 1}{3}\,.$$

Assume that we have 100 cycles. After the k-th cycle, the amount of money used is M_k and the amount added to the bag is W_k. We then have

$$W_k = 2 \times \frac{2M_k + 1}{3} \quad \text{and} \quad M_1 + \cdots + M_{100} = 2500\,.$$

Hence, we solve for the amount of money in the bag after the 100-th win:

$$W_1 + \cdots + W_{100} = \frac{4}{3}(M_1 + \cdots + M_{100}) + 100 \times \frac{2}{3} = \frac{4}{3} \times 2500 + 100 \times \frac{2}{3} = 3400\,.$$

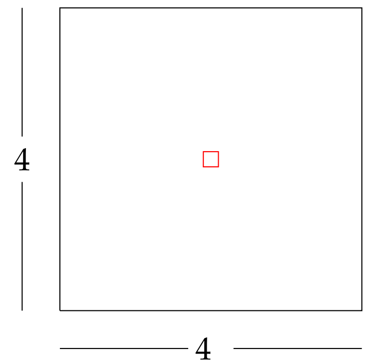
□

Solution A4.
Condition (b) implies that every row has every remainder $0, \ldots, 8$ when divided by 9 and each remainder appears exactly once in every row. Hence, the table has nine appearances of remainder 0, nine appearances of remainder 1 and so on. Condition (a) implies that off diagonal remainders come in pairs. Since the total number of integers with the same remainder is odd, one remainder should be on the main diagonal. Hence, the main diagonal has every remainder $0, \ldots, 8$, and so the sum of the remainders on the main diagonal is

$$0 + 1 + \cdots + 8 = 9 \times 4$$

which is divisible by 9. □

Solution A5.
The length of the red square below is $4 - 3.8 = 0.2$.

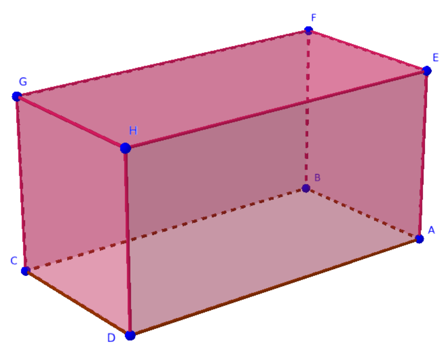


If the centre of the coin is within the red square, then the coin will lie completely inside the tile. The probability that coin lies within the tile is

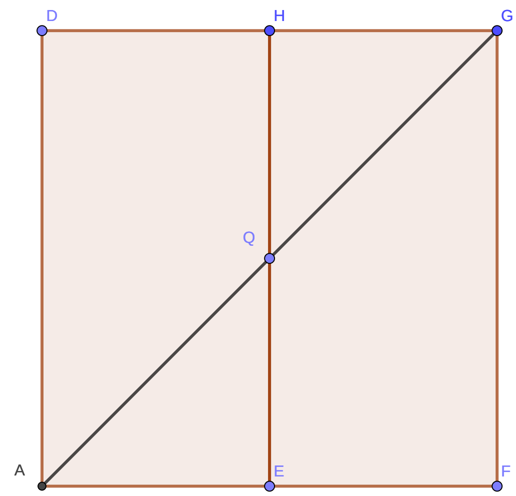
$$\frac{\text{Area of red square}}{\text{Area of tile}} = \frac{0.2^2}{4^2} = \frac{1}{400}.$$

$\square$

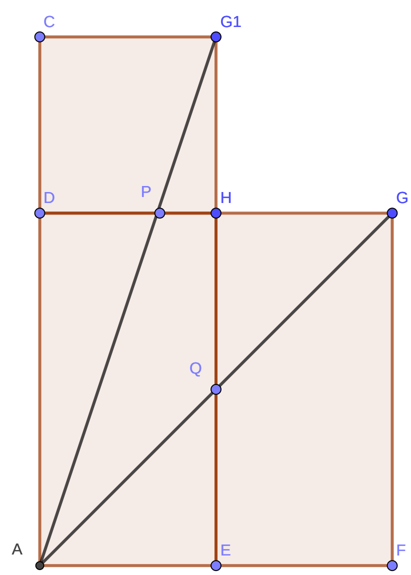
Solution A6.
Label the vertices of the box as on the diagram and assume that the bug is at the vertex A. We need to find the shortest path to the vertex G.



If the bug travels across the edge EH, then, after unfolding the faces $AEHD$ and $EHGF$, the shortest path across this edge looks as follows:



The length of such a path is
$$\sqrt{2^2 + 2^2} = \sqrt{8}\,.$$
Another possible travel is across the edge HD. In this case, after unfolding the faces $AEHD$ and $DHGC$, the shortest path is as follows:



The length of such a path is
$$\sqrt{3^2 + 1^2} = \sqrt{10}\,.$$
Hence the shortest path is through the middle point of the edge EH. $\square$

B Senior Division – Problems

Solution B1.
The first player wins every time by using the following strategy. On the first move, he draws the segment which splits the points into 9 and 9 on each side of the segment. On subsequent moves, the first player mirrors each move of the second player with respect to the segment above. Such approach ensures that there is always a move available for the first player. □

Solution B2.
Label the bags $1, 2, \ldots, 10$. Choose one coin from the bag 1, two coins from the bag 2, $\ldots$, ten coins from the bag 10. Weigh all of the chosen coins on the scale and assume that the reading is W. If all the coins where genuine, then the total weight is

$$1 \times 10 + 2 \times 10 + \ldots + 10 \times 10 = 550g\,.$$

However, if the bag N has all counterfeit coins, then

$$N = W - 550\,.$$

□

Solution B3.
Note that if the gambler starts with bet \$1, experiences $n-1$ losses and wins on n-th turn, then the amount of money used is, in dollars,

$$M = 1 + 3 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}\,.$$

At the end of this cycle, the gambler adds to the bag $W = 2 \times 3^{n-1}$ dollars. That is,

$$W = 2 \times \frac{2M+1}{3}\,.$$

Assume that we have 100 cycles. After the k-th cycle, the amount of money used is M_k and the amount added to the bag is W_k. We then have

$$W_k = 2 \times \frac{2M_k + 1}{3} \quad \text{and} \quad M_1 + \cdots + M_{100} = 2500\,.$$

Hence, we solve for the amount of money in the bag after the 100-th win:

$$W_1 + \cdots + W_{100} = \frac{4}{3}(M_1 + \cdots + M_{100}) + 100 \times \frac{2}{3} = \frac{4}{3} \times 2500 + 100 \times \frac{2}{3} = 3400\,.$$

□

Solution B4.
Look at the diagram below. Using the right angled triangle $\triangle ADG$, we see that

$$AB = 2R\sin\frac{\beta}{2}\,.$$

Similarly,

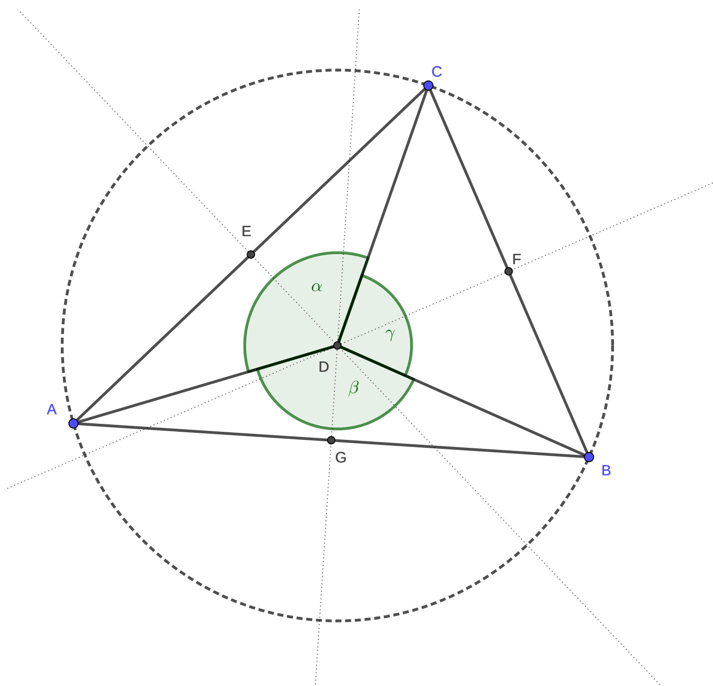
$$AC = 2R\sin\frac{\alpha}{2} \quad \text{and} \quad BC = 2R\sin\frac{\gamma}{2}\,.$$

Hence, we need to prove that

$$\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \geq 2\,,$$

where

$$0 \leq \alpha, \beta, \gamma \leq \pi \quad \text{and} \quad \alpha + \beta + \gamma = 2\pi\,.$$



We restate the problem as follows: prove that

$$\sin x + \sin y + \sin z \geq 2\,,$$

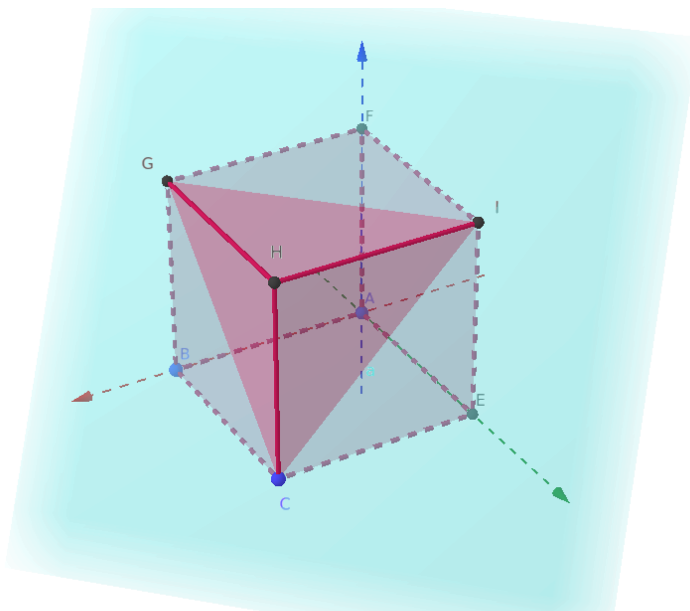
where

$$x + y + z = \pi \quad \text{and} \quad 0 \leq x, y, z \leq \frac{\pi}{2}\,.$$

The set

$$\Omega = \{(x, y, z): \quad 0 \le x, y, z \le \frac{\pi}{2} \quad \text{and} \quad x + y + z = \pi\}$$

is pictured on the following diagram. It is a triangle.



We note that the function

$$h \mapsto \sin h, \quad h \in [0, \frac{\pi}{2}]$$

is concave. Hence, the function

$$f: (x, y, z) \mapsto \sin x + \sin y + \sin z, \quad (x, y, z) \in \Omega$$

is concave.

We also note that a concave function on a convex polygon attains its minimum at the vertices of the polygon. Hence, we conclude that the function f attains its minimum at either points G, I or C on the diagram. Note that

$$f(G) = f(C) = f(I) = f(\frac{\pi}{2}, \frac{\pi}{2}, 0) = 2\,.$$

Hence, the desired inequality is proved. $\square$

Solution B5.
The following diagram describes the different states of the tie-breaker game. We show the score in the form $A : B$ where the first number stands for the score of the player A and the second number stands for the score of the player B. The '+' symbol shows who is to serve.

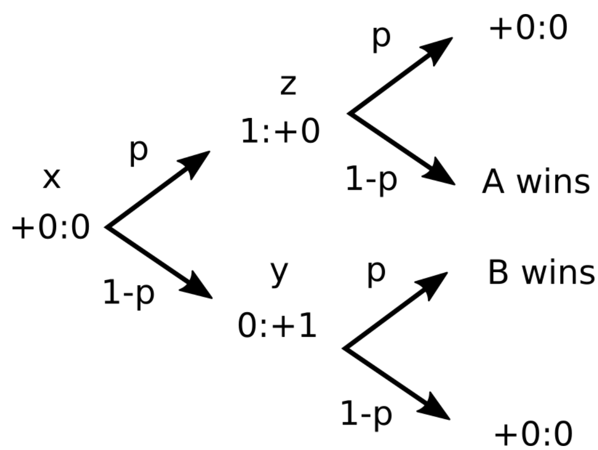
Assume that x is the probability for server to win the game from the state $+0 : 0$; y is the probability for server to win the game from the state $+1 : 0$; and z is the probability of server to win the game from the state $+0 : 1$. We need to find $1 - y$.

The diagram suggests the following relations:

$$\begin{cases} x = p(1-z) + (1-p)(1-y) \\ y = p + (1-p)(1-x) \\ z = p(1-x) \end{cases} .$$

Solving this system of linear equations, we find:

$$x = \frac{1}{2}, \quad y = \frac{1}{2} + \frac{p}{2}, \quad z = \frac{p}{2}.$$



□