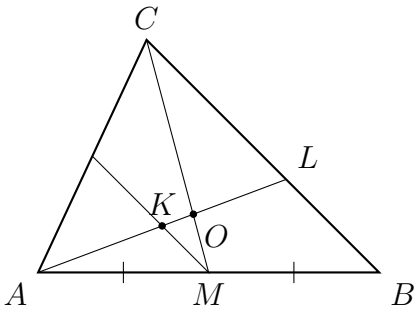
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 9, July 1, 2019

1. Find the sum of all n-digit long numbers formed by $1, 2, 3, \ldots, n$. For example, if $n = 3$ then the sum of all 3-digit long numbers is $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

2. Evaluate $\sqrt[4]{2} \times \sqrt[8]{4} \times \sqrt[16]{8} \times \sqrt[32]{16} \times \sqrt[64]{32} \ldots$.

3. Several positive integers are written on a blackboard. The sum of any two of them is some power of two (for example, 2, 4, 8,...). What is the maximal possible number of different integers on the blackboard?

4. For a triangle ABC, M is the midpoint of the side AB and L is some point along the side BC. Let O be the point of intersection between the lines LA and MC, and let K be the point of intersection between LA and the line passing through M, parallel to BC, as shown in the diagram below.



NOT TO SCALE

   (a) Show that the triangles $\triangle KMO$ and $\triangle LCO$ are similar.

   (b) Suppose the length LA is twice as long as MC, and $\angle OLC = 45°$. Prove LA is perpendicular to MC.

5. Consider the polynomial $p(x) = x^4 + 37x^3 + 71x^2 + 18x + 3$. If a, b, c and d are roots of $p(x)$, find a polynomial whose roots are $\dfrac{abc}{d}, \dfrac{acd}{b}, \dfrac{abd}{c}$ and $\dfrac{bcd}{a}$.

Senior Questions

1. The speed of a multi-pole DC motor is described by the following simplified relation:

$$j\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} = I_{motor}, \qquad (1)$$

where θ is the angular position of the motor, j is the rotational inertia of the load, c is the damping coefficient of the load and I_{motor} is the current through the motor circuit.

Additionally, the motor circuit has an associated resistance (R) and back EMF ($k_m\frac{d\theta}{dt}$) described by the following formula:

$$V_{in} = R \times I_{motor} + k_m\frac{d\theta}{dt}, \qquad (2)$$

where V_{in} is the applied battery voltage. Given $j = 5$, $c = 1$, $V_{in} = 12$ volts, $R = 10$ ohms, $k_m = 5$, find:

   (a) The limiting angular speed of the motor.

   (b) The initial acceleration of the motor from a stationary state.

   (c) Solve for $\theta(t)$.

2. Using straight-edge and compass techniques, describe how to construct a triangle given its circumcircle ($\mathcal{C}$) and the three points on it at which the altitude (A), the angle bisector (B), and the median (C) drawn from the same vertex intersect the circle.