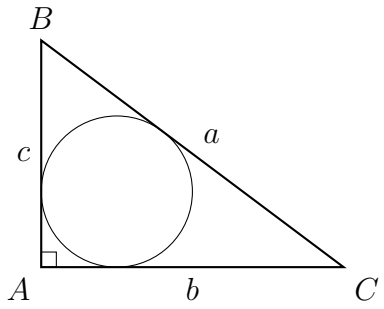
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 7, June 17, 2019[1]

1. How many pairs of integers, x and y, are there such that $x^2 - y^2 = 2019$?

2. Let $\triangle ABC$ be a triangle with sides of length a, b and c and $\angle A = 90°$. The *incircle* of a triangle is a circle inside the triangle that is tangent to each side. (See diagram below.) Find the radius of the incircle of $\triangle ABC$ in terms of a, b and c.



3. Let N be a number of the form $N = \underbrace{333\ldots333}_{61\times 3's}$, and M a number of the form $M = \underbrace{666\ldots666}_{62\times 6's}$. Find $N \times M$.

4. Let x be a positive odd number, and a a positive integer greater than 2. If a^x has remainder r_1 when divided by $(a-1)$ and r_2 when divided by $(a+1)$, find $r_1 + r_2$.

5. Let $[x]$ denotes the greatest integer less than or equal to x, where x is some real number. How many positive integers less than 1001 can be expressed in the form $[2x] + [4x] + [6x] + [8x]$?

6. On a bicycle, tyre wear is proportional to the distance travelled, front tyres lasting x kilometres and rear tyres lasting y kilometres $(x < y)$. An advertisement claims that a set of tyres lasts at least $(x+y)/2$ kilometres provided you interchange front and rear tyre after an appropriate distance. Investigate.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

Senior Questions

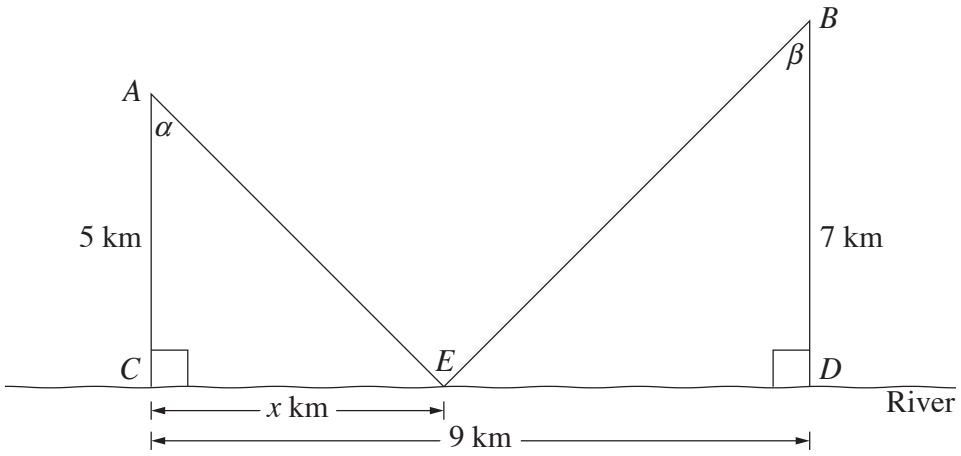
1. Consider the quadratic equation

$$f(x) = x^2 - 2(c+1)x + c - 3,$$

where c is some real number. Let $\alpha, \beta > 0$, and suppose $\alpha + \frac{1}{\alpha}$ and $2 - \beta - \frac{1}{\beta}$ are the roots of $f(x)$. Find all possible values for c.

2. Given that a, b and c are positive integers, find the conditions for which the equation $\sqrt{a} - b = \sqrt{c}$ has a solution.

3. The following is Question 16(a) from the 2017 HSC Mathematics paper. (You don't have to work through the question, but you should read it.)

> To get some exercise, John cycles from home directly to point E on the river, x km from C, before cycling directly to school at B, as shown in the diagram.
>
> 
>
>
> The total distance John cycles from home to school is L km.
>
> (i) Show that $L = \sqrt{x^2 + 25} + \sqrt{49 + (9-x)^2}$.
>
> (ii) Show that if $\dfrac{dL}{dx} = 0$, then $\sin\alpha = \sin\beta$.
>
> (iii) Find the value of x that makes $\sin\alpha = \sin\beta$.
>
> (iv) Explain why this value of x gives a minimum for L.

In this question, calculus is used to find the shortest length of the path A–E–B. There is a much simpler method for solving this problem that does not require the use of calculus. See if you can figure out what it is.