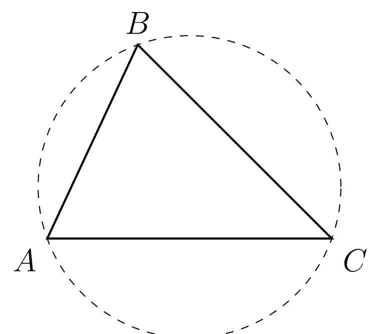
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 6, June 10, 2019[1]

1. Some simple construction problems using straight-edge and compass techniques:

   (a) Given an interval AB, describe how to construct an equilateral triangle with AB as a base.

   (b) Given an triangle ABC, describe how to construct its circumcircle. (The circumcircle is the unique circle which passes through the three vertices of the triangle.)



2. Find the number of ordered pairs (x, y) of non-negative integers such that $x+y \leq 100$.

3. Let p be your favourite prime number greater than 100, and a, b positive integers such that $p^2 + a^2 = b^2$. Find $\frac{a+b}{p}$.

4. At a party of 21 people each person knows at most four others. Prove that there are five in the party who mutually do not know each other.

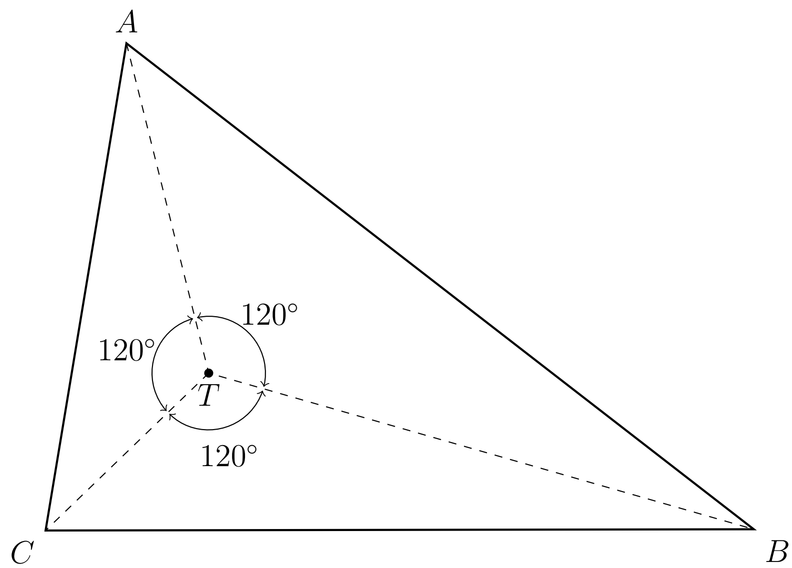
5. Let $f(x)$ be a polynomial with integer coefficients. Suppose a_1, a_2, a_3, a_4, a_5 are distinct integers such that $f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 2015$. Find the number of integral solutions for the equation $f(x) = 2016$.

6. M is the midpoint of the side CA of triangle ABC. P is some point on the side BC. AP and BM intersect at the point O. If $BO = BP$, determine $\frac{|OM|}{|PC|}$.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

Senior Questions

1. Suppose that ABC is a triangle in which all internal angles are less than $120°$. The Fermat-Torricelli point of $\triangle ABC$, shown as T in the diagram below, is the point inside the triangle such that $\angle ATB = \angle ATC = \angle BTC = 120°$.



Given $\triangle ABC$, but not the location of its Fermat-Torricelli point, can you describe a method using straight-edge and compass techniques to find the point T?

Note: The restriction on the size of the internal angles of $\triangle ABC$ is simply to ensure that T lies inside the triangle. It does not have any bearing on the construction.

2. Let $z = \cos\theta + i\sin\theta$ be a fifth root of unity other than one.

   (a) If $-\pi < \theta \leq \pi$, what are the possible values of θ?

   (b) Explain why z is a solution to

   $$z^4 + z^3 + z^2 + z + 1 = 0. \qquad (*)$$

   (c) If $x = z + \frac{1}{z}$, show that $x = 2\cos\theta$.

   (d) Show that $x^2 + x - 1 = 0$.
   Hint: First divide (*) by z^2.

   (e) Hence find an expression for the exact values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$.