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**MATHEMATICS ENRICHMENT CLUB.**
**Problem Sheet 5, June 3, 2019**[1]

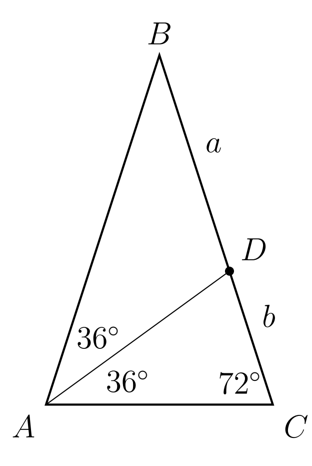
1. An arithmetic sequence consists of 11 terms, and the sum of the terms equals 220. Find the middle term in the sequence.

2. (a) At a birthday party, the number of guests Bernard invited is equal to his age. During the party, each person shakes hands with every other person exactly once. If there are a total of 253 handshakes during the party, how old is Bernard?

   (b) In the middle of the party, Bernard arranges his guests and himself to be seated at two round tables. If Bernard would like the number of people sitting at each table to be as even as possible, in how many ways can he do this (two arrangements are counted as the same if they can be obtained by rotating the tables)?

3. Prove that seven is the only prime number that is one less than a perfect cube.

4. Let $\triangle ABC$ be a isosceles triangle with $\angle A = \angle C = 72°$. The point $D$ lies on $BC$ such that $AD$ bisects $\angle BAC$, as shown in the diagram below.




   (a) Let the lengths of $BD$ and $CD$ be $a$ and $b$, respectively. Find $\frac{a}{b}$.

   (b) Express $\cos(36°)$ in terms of $a$ and $b$, and thus using part $(a)$, find an exact expression for $\cos(36°)$.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

5. The number 4 has an odd number of odd positive divisors, namely 1, and an even number of even positive divisors, namely 2 and 4. Is there a number with an odd number of even positive divisors and an even number of odd positive divisors?

6. $a$ and $b$ are positive integers. Of the following statements, three are true, one is false.

   (a) $a+1$ is divisible by $b$.

   (b) $a = 2b+5$.

   (c) $a+b$ is divisible by 3.

   (d) $a+7b$ is a prime number.

   Determine the possible values of $a$ and $b$.

**Senior Questions**

1. (a) Let $n$ be a positive integer. Prove that

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \ldots + a + 1),$$

   for all real numbers $a$.

   (b) You are given that $n \geq 2$ and $a^n - 1$ is prime. Prove that $a = 2$, and $n$ must be prime.

2. Let $F_n$ be the $n$th Fibonacci number. The Fibonacci numbers, $0, 1, 1, 2, 3, 5, \ldots$, can be defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

   with initial conditions $F_0 = 0$ and $F_1 = 1$.

   (a) Suppose that $F_n$ has the form $F_n = Ar^n$ for some constants $A$ and $r$. Show that this implies that

$$r^2 - r - 1 = 0.$$

   This equation is called the *characteristic equation* of the recurrence.

   (b) There are two distinct solutions of the characteristic equation, namely $\phi_+ = \frac{1+\sqrt{5}}{2}$ and $\phi_- = \frac{1-\sqrt{5}}{2}$. This means that

$$F_n = \alpha(\phi_+)^n + \beta(\phi_-)^n,$$

   where $\alpha$ and $\beta$ are two constants yet to be determined.

   Use the two initial conditions to find the values of $\alpha$ and $\beta$, and hence determine a closed formula for $F_n$.

   (c) The Lucas numbers $(L_n)$ follow the same recurrence as the Fibonacci numbers, but with the initial conditions $L_0 = 2$ and $L_1 = 1$. Find a closed formula for $L_n$.