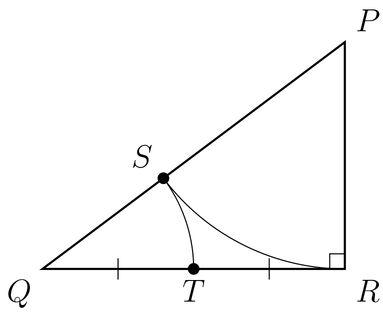
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 15, September 10, 2019

1. *AMC 2012 Senior Division, Q12.*
   Triangle PQR is right-angled at R. The circle with centre P and radius PR cuts PQ at S and the circle with centre Q and radius QS cuts QR at T.

   If T bisects QR, find the ratio $QS : SP$.

2. Find all possible solutions to
   $$\frac{x}{y} + \frac{1}{x} + \frac{1}{y} = \frac{1}{4}$$
   if x and y are positive integers.

3. Explain why a number made up of the same digit can only be prime if the digit is one and the number of digits is itself prime.

4. Find the smallest positive integer that is a multiple of 9 and has no odd digits.

5. *Adapted from AMC 2012 Senior Division, Q21.*
   Let $p(x)$ be a polynomial such that
   $$p(x) = (x-2)^{2019}(x+2019) + (x-2)^{2018}(x+2018) + \ldots + (x-2)(x+1).$$
   Find the sum of the coefficients of $p(x)$.

6. Let m and n be positive integers. Find the number of ordered pairs (m, n) such that the expression $(m-8)(m-10) = 2^n$ is satisfied.

Senior Questions

1. The triangular numbers are given by $T_n = 1 + 2 + \cdots + n$ for n a positive integer ($T_1 = 1$).

   Discover and prove a formula for
   $$T_n \left(\frac{1}{T_1} + \frac{1}{T_2} + \ldots + \frac{1}{T_n} \right).$$

2. A, B, C and D are points on the parabola $y = x^2$ such that AB and CD intersect on the y-axis. Determine the x-coordinate of D in terms of the x-coordinates of A, B and C, which are a, b and c respectively.