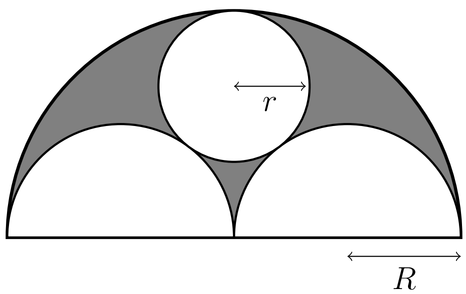
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13, August 27, 2019

1. Suppose that x is a three-digit positive integer. The six-digit number y is created by repeating the digits of x. If y is also a multiple of x^2, find the values of x and y.

2. Find the last digit of $1^5 + 2^5 + \ldots + 2019^5$.

3. Two congruent semi-circles of radius R are drawn inside a larger semicircle. A smaller circle, with radius r, is also inscribed in the larger semi-circle so that it is tangent to all three semicircles, as shown in the diagram below.



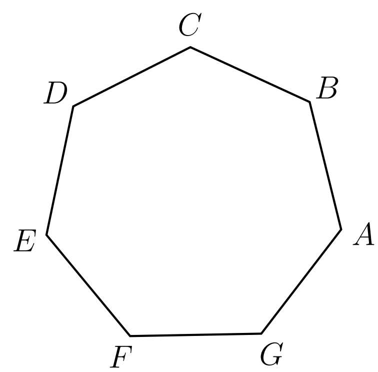
   Show that $R : r = 3 : 2$.

4. Integers $1, 2, \ldots, 100$ are written in a circle, not necessarily in that order. Can it be that the absolute value of the difference between any two adjacent integers is at least 30 and at most 50?

5. Consider an arbitrary number $a > 0$. We know that the inequality $10 < a^x < 100$ has exactly 5 positive integer solutions for x. How many solutions in positive integers may the inequality $100 < a^x < 1000$ have? In each case, list the solutions.

Senior Questions

1. Suppose that $ABCDEFG$ is a regular heptagon, as shown in the diagram.



Show that

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

2. Seventeen primes $p_1 < p_2 < \ldots < p_{17}$ have the property that the sum of their squares is also a square. Prove that $p_{17}^2 - p_{16}^2$ is divisible by p_1.

3. Twelve knights $k_1, k_2, \ldots, k_{12}$ are seated in anti-clockwise order around a circular table. What is the minimal number of swaps required to change their order to a clockwise one, if any swap can be made only between adjacent knights? What is the answer if there are thirteen knights?