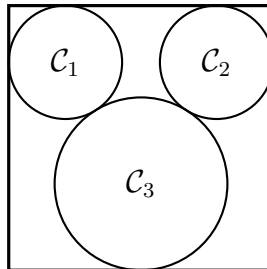**MATHEMATICS ENRICHMENT CLUB.**
Problem Sheet 11, August 13, 2019

1. Find all positive integers n, such that

$$\frac{n^2 + 11n + 2}{n + 5},$$

   is also an integer.

2. Three circles fit inside a square as shown in the diagram. The two smaller circles, $\mathcal{C}_1$ and $\mathcal{C}_2$, have radius 3 and each is tangent to the larger circle, $\mathcal{C}_3$. The square has side length 14. Find the radius of $\mathcal{C}_3$.

3. Find all prime numbers, p, such that $4p^2 + 1$ and $6p^2 + 1$ are both prime.

4. Each of 100 stones has a sticker showing its true weight. No two stones weigh the same. Mischievous Greg wants to rearrange stickers so that the sum of the numbers on the stickers for any group containing from 1 to 99 stones is different from the true weight of this group. Is this always possible?

5. We have 99 locked boxes and 99 keys, each opening just one lock. Each box has an opening through which we throw one key at random.

   (a) We break open one of the boxes. What is the probability that we can open all the other boxes without breaking them?

   (b) If we initially break two boxes instead of one, what is the probability that we can open all the other boxes without breaking them?
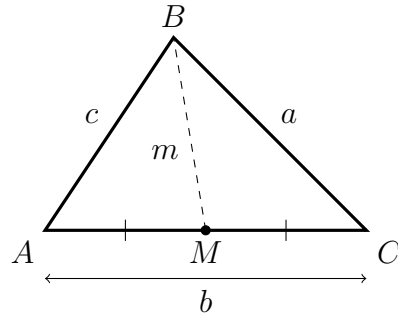
Senior Questions

1. (a) A parallelogram has sides of length a and b and diagonals of length x and y, as shown in the diagram.
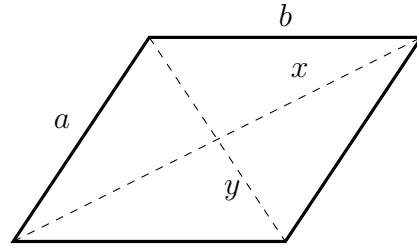
   Prove the parallelogram law. That is, show that

   $$x^2 + y^2 = 2a^2 + 2b^2.$$

   (b) The triangle ABC, with sides of length a, b and c, is shown in the diagram. The interval MB is a median of $\triangle ABC$ with length m.

   Find an expression for m in terms of a, b, and c. In addition, show that if $\triangle ABC$ is equilateral with side length s, then $m = \frac{\sqrt{3}}{2}s$.

2. Let T be a set of integers with the following properties:

   (a) T contains $0, 1, 3, 4, 5$.

   (b) If a, b, c, d are distinct elements of T such that

   $$a + b = c + d,$$

   then
   $$r = -(a + b) = -(c + d)$$

   is also in T. For example $0 + 4 = 1 + 3$; therefore -4 is also an element of T.

   Prove that T contains all integers.