MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 10, August 6, 2019

1. Alice and Carla are playing a dice game. Here's how it works:

   - Each person rolls a die, and the highest number rolled of the two is recorded.
   - If the highest number rolled is a $1, 2, 3$ or 4, Alice wins.
   - If the highest number rolled is a 5 or a 6, Carla wins.

   On average, who is more likely to win: Alice, Carla, or are the probabilities equal?

2. How many 3 digit positive integers are the sum of exactly 9 distinct powers of 2?

3. Given that $a + b = 1$ and $a^2 + b^2 = 2$, what is the value of $a^7 + b^7$?

4. Given that x and y are distinct, non-negative real numbers such that

$$x + \sqrt{y} = y + \sqrt{x},$$

   determine the maximum value of $x + y$.

5. Let ABC be a triangle, and let D, E and F be the feet of the altitudes of $\triangle ABC$, as shown in the diagram below. (An altitude is the perpendicular from a vertex to the opposing side.) The points D, E and F form another triangle which is called the *orthic triangle* of $\triangle ABC$.
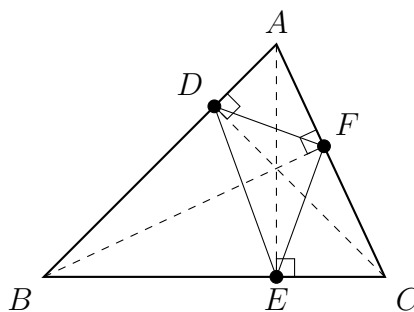
   (a) Show that $\triangle EFC$ is similar to $\triangle ABC$.
   (b) Show that the altitudes of $\triangle ABC$ are the angle bisectors of the orthic triangle, $\triangle DEF$.

Senior Questions

1. Find the remainder when x^{2019} is divided by $x^2 - 1$.

2. In $\triangle ABC$, $AC = 6$ cm, $BC = 4$ cm, $\angle A = \theta$ and $\angle B = 2\theta$, as shown below.
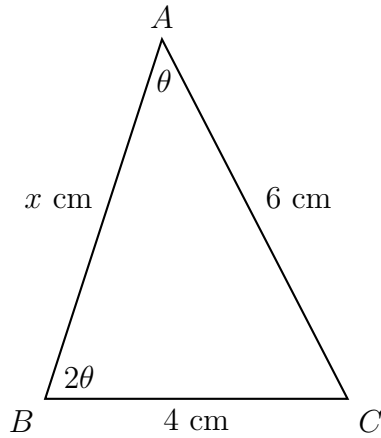
Determine the value of x.

3. Find all solutions of $2^x + 3^x + 6^x = x^2$.