MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 18, October 1, 2018

1. Each tile has to cover one white and one black square, pairing up the squares into black-white couples. Removing opposite corners removes two squares of the same colour, so there are no longer equal numbers of black and white squares. Thus the tiling can't be done.

2. Suppose that the three digit number has the form abc where $a \in \{1, 2, \ldots, 9\}$ and $b, c \in \{0, 1, 2, \ldots, 9\}$. Then are trying to solve

$$100a + 10b + c = a! + b! + c! \quad (*).$$

Firstly, note that

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \\ 6! &= 720 \\ 7! &= 5040 \end{aligned}$$

As $7! = 5040$ is larger than 3 digits, none of 7, 8 or 9 may be used. Also, $6! = 720$ and since our number cannot contain 7, 8 or 9, that means that 6 is also not possible. This gives an upper limit for the size of number of $5! + 5! + 5! = 360$. Also, note that $4! + 4! + 4! = 72 < 100$, so we need to have at least one 5.

Thus far, we know that $a \in \{1, 2, 3\}$, we must have at least one 5, and one other digit, $x \in \{0, 1, 2, 3, 4, 5\}$. If $a = 1$, for the RHS of (*), we have

$$\begin{aligned} 1! + 5! + 0! &= 1 + 120 + 1 = 122 \\ 1! + 5! + 1! &= 1 + 120 + 1 = 122 \\ 1! + 5! + 2! &= 1 + 120 + 2 = 123 \\ 1! + 5! + 3! &= 1 + 120 + 6 = 127 \\ 1! + 5! + 4! &= 1 + 120 + 25 = 145 \\ 1! + 5! + 5! &= 1 + 120 + 120 = 241 \end{aligned}$$

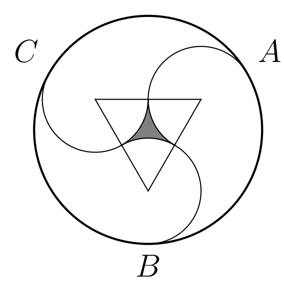
We can see that the only option that works here is 145.

If $a = 2$, we have

$$
\begin{aligned}
2! + 5! + 0! &= 2 + 120 + 1 = 123 \\
2! + 5! + 1! &= 2 + 120 + 1 = 123 \\
2! + 5! + 2! &= 2 + 120 + 2 = 124 \\
2! + 5! + 3! &= 2 + 120 + 6 = 128 \\
2! + 5! + 4! &= 2 + 120 + 25 = 147 \\
2! + 5! + 5! &= 2 + 120 + 120 = 242
\end{aligned}
$$

None of these work. That leaves us with $a = 3$, but in this case $3! + 5! + 5! = 6 + 120 + 120 = 246 < 300$. Hence the only number is 145.

3. This is closely related to Question (4) from last week. The simplest way to find the area of the tear-drop shape is to calculate the shaded area in the centre of the circle. Since the three tear-drops are congruent, if we subtract the shaded area from the area of the circle and the divide the result by three, we will have the area of each tear-drop.



We can connect the centres of the three arcs to form an equilateral triangle with side length $2r$. The area of this triangle is given by

$$\frac{1}{2}(2r)^2 \sin 60^\circ = \sqrt{3}r^2.$$

Each arc cuts out a sector of area $\frac{1}{6} \times \pi r^2$ from the triangle. Thus the area of the shaded region is

$$\sqrt{3}r^2 - \frac{\pi r^2}{2} = \frac{(2\sqrt{3} - \pi)r^2}{2}.$$

Recall that from last week that $R = r\left(1 + \frac{2}{\sqrt{3}}\right)$, so the area of the circle in terms of r is

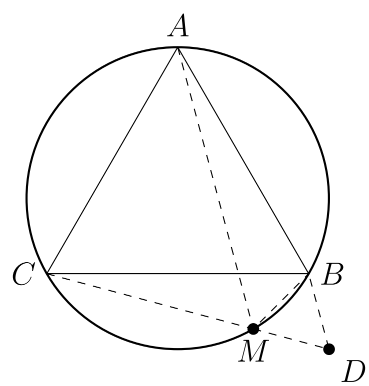
$$
\begin{aligned}
\pi\left[r\left(1 + \frac{2}{\sqrt{3}}\right)\right]^2 &= \pi r^2 \left(\frac{\sqrt{3}+2}{\sqrt{3}}\right)^2 \\
&= \frac{(7 + 4\sqrt{3})\pi r^2}{3}.
\end{aligned}
$$

If T is the area of one tear-drop, then

$$\begin{aligned} T &= \frac{1}{3}\left[\frac{(7+4\sqrt{3})\pi r^2}{3} - \frac{(2\sqrt{3}-\pi)r^2}{2}\right] \\ &= \frac{r^2}{3}\left[\frac{(7+4\sqrt{3})\pi}{3} - \frac{(2\sqrt{3}-\pi)}{2}\right] \\ &= \frac{r^2}{3}\left[\frac{2(7+4\sqrt{3})\pi - 3(2\sqrt{3}-\pi)}{6}\right] \\ &= \frac{r^2}{3}\left[\frac{(17+8\sqrt{3})\pi - 6\sqrt{3}}{6}\right]. \end{aligned}$$

4. For a game to be a draw, all 9 squares must be filled without anyone winning along the way. Now a game is not just the configuration at the end, but the order the O's and X's get put there. So count the number of ways of arranging the X's and O's so that there are no three-in-a-rows (there are 16), then for each of these arrangements, count the number of ways of playing the game to get there ($5! \times 4!$).

5. Extend the interval CM to D so that $DM = MB$. Then $CD = CM + MB$.



Since $MBAC$ is a cyclic quadrilateral and $\angle BAC = 60°$, $\angle CMB = 120°$. Hence $\angle BMD = 60°$. Thus $\triangle BDM$ is an equilateral triangle, and so $MB = BD$. Furthermore, since $\triangle ABC$ is equilateral, $AB = BC$, and since $\angle MBD = \angle ABC = 60°$, $\angle CBD = \angle ABM$. Thus $\triangle ABM \equiv CBD$ by SAS. Thus $AM = CD$, as they are corresponding sides in congruent triangles. Consequently, $AM = MB + MC$, as required.

Senior Questions

1. (a) From the given definition,

$$\begin{aligned}\langle p_0, p_1\rangle &= \int_{-1}^{1} 1\times x\,dx \\ &= \left[\frac{x^2}{2}\right]_{-1}^{1} \\ &= \frac{1}{2}-\frac{1}{2} = 0.\end{aligned}$$

(We can also use symmetry properties of an odd function to obtain this result.)

(b) We want

$$\begin{aligned}\langle \alpha_0 p_0, \alpha_0 p_0\rangle &= 1 \\ \int_{-1}^{1} (\alpha_0)^2\,dx &= 1 \\ (\alpha_0)^2\,[x]_{-1}^{1} &= 1 \\ 2(\alpha_0)^2 &= 1 \\ \alpha_0 &= \frac{1}{\sqrt{2}}.\end{aligned}$$

And

$$\begin{aligned}\langle \alpha_1 p_1, \alpha_1 p_1\rangle &= 1 \\ \int_{-1}^{1} (\alpha_1)^2 x^2\,dx &= 1 \\ (\alpha_1)^2\left[\frac{x^3}{3}\right]_{-1}^{1} &= 1 \\ \frac{2}{3}\times(\alpha_1)^2 &= 1 \\ \alpha_1 &= \sqrt{\frac{3}{2}}.\end{aligned}$$

(c) Suppose that $q_2 = a_2x^2 + a_1x + a_0$, where a_0, a_1 and a_2 are to be determined. If $\langle q_2, q_0\rangle = 0$, then

$$\begin{aligned}\int_{-1}^{1} \alpha_0(a_2x^2 + a_1x + a_0)\,dx &= 0 \\ \therefore \int_{-1}^{1} (a_2x^2 + a_1x + a_0)\,dx &= 0.\end{aligned}$$

By symmetry, $\int_{-1}^{1} a_1x\,dx = 0$ for any value of a_1.

Similarly,

$$\begin{aligned}\int_{-1}^{1}(a_2x^2+a_0)\,dx &= 2\int_{0}^{1}(a_2x^2+a_0)\,dx\\ &= 2\left[\frac{a_2x^3}{3}+a_0x\right]_0^1\\ &= 2\left(\frac{a_2}{3}+a_0\right).\end{aligned}$$

Thus $a_2=-3a_0$.
If $\langle q_2,q_1\rangle=0$, then

$$\begin{aligned}\int_{-1}^{1}\alpha_1x(a_2x^2+a_1x+a_0)\,dx &= 0\\ \alpha_1\int_{-1}^{1}(a_2x^3+a_1x^2+a_0x)\,dx &= 0.\end{aligned}$$

By symmetry, $\int_{-1}^{1}(a_2x^3+a_0x)\,dx = 0$, for any choice of a_2 or a_0, however, $\int_{-1}^{1}a_1x^2\,dx=0$, only if $a_1=0$.
Now $\langle q_2,q_2\rangle=1$, so

$$\begin{aligned}\int_{-1}^{1}(a_0-3a_0x^2)(a_0-3a_0x^2)\,dx &= 1\\ (a_0)^2\int_{-1}^{1}(1-3x^2)(1-3x^2)\,dx &= 1\\ (a_0)^2\int_{-1}^{1}1-6x^2+9x^4\,dx &= 1\\ 2(a_0)^2\int_{0}^{1}1-6x^2+9x^4\,dx &= 1\\ 2(a_0)^2\left[x-2x^3+\frac{9x^5}{5}\right]_0^1 &= 1\\ \frac{8(a_0)^2}{5} &= 1.\end{aligned}$$

So $a_0=\sqrt{\frac{5}{8}}$, and $p_2(x)=\sqrt{\frac{5}{8}}(1-3x^2)$

(d) We take the inner product of both sides of

$$p(x)=\beta_0q_0(x)+\beta_1q_1(x)+\beta_2q_2(x),$$

with q_0, q_1 and q_2 successively to show that $\beta_0=\langle p(x),q_0\rangle$, $\beta_1=\langle p(x),q_1\rangle$, and $\beta_2=\langle p(x),q_2\rangle$.