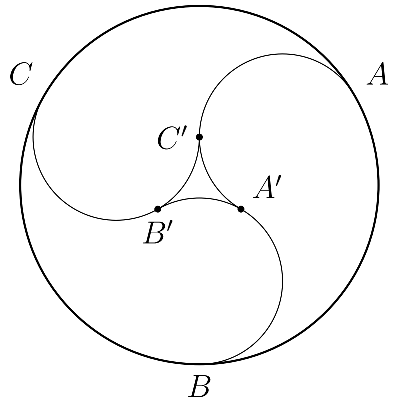
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 18, September 24, 2018

1. A chess board is an 8×8 grid of squares coloured white or black so that no two adjacent squares are the same colour. Given tiles that are 2×1 grid squares, it is possible to cover the chessboard completely, and it takes precisely 32 tiles. Show that it is impossible to cover a chessboard with opposite corners removed.

2. Find all 3 digit numbers which are equal to the sum of the factorials of their digits.

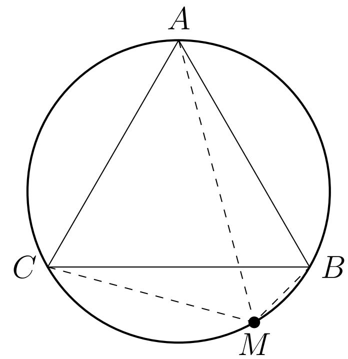
3. In the diagram below, ABC is a circle of radius R with 3 tear-drop shapes inside. Each of the arcs $AC'A'$, $BA'B'$ and $CB'C'$ are of circles of the same radius, r. Find the area of each tear drop in terms of r.



4. Tic-tac-toe is a game played by two players who take turns marking either X or O in a square on a 3×3 grid. A player wins if they get 3 of their symbols in a row, but if the grid is filled without a winner the game is a draw.

   How many tic-tac-toe games end in a draw?

5. The point M lies on the circumcircle of the equilateral triangle $\triangle ABC$, as shown in the diagram.



Prove that $AM = MB + MC$.

Senior Questions

1. Let f and g be real-valued, continuous functions defined on $-1 \leq x \leq 1$. We define the *inner product* of f and g, $\langle f, g\rangle$, as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Consider the polynomials $p_0(x) = 1$ and $p_1(x) = x$.

(a) Two functions, f and g, are said to be *orthogonal* if $\langle f, g\rangle = 0$. Show that p_0 and p_1 are orthogonal.

(b) A function f is said to be *normalized* if $\langle f, f\rangle = 1$. Find factors α_0 and α_1 such that the polynomials $q_0 = \alpha_0 p_0$ and $q_1 = \alpha_1 p_1$ are not only orthogonal but also normalized.

(c) A set of polynomials that are all normalized and mutually orthogonal is called *orthonormal.* Find a quadratic q_2 so that $\{q_0, q_1, q_2\}$ form an orthonormal set.

(d) Show that any polynomial of degree 2 or less can be written as a linear combination of q_0, q_1 and q_2. That is, show that there are constants β_0, β_1 and β_2 such that

$$p(x) = \beta_0 q_0(x) + \beta_1 q_1(x) + \beta_2 q_2(x),$$

for any arbitrary polynomial $p(x) = ax^2 + bx + c$.