Parabola Volume 54, Issue 2 (2018)

57th Annual UNSW School Mathematics Competition: Competition Problems

Solutions by Denis Potapov[1]

A Junior Division – Problems

Problem A1:
Prove that a positive integer has an odd number of divisors if and only if it is the square of another integer.

Problem A2:
Is it possible to draw five straight lines on the plane such that every line intersects exactly three other lines?

Problem A3:
You are given six coins and you know that two of the coins are counterfeit. You also know that the counterfeit coins are lighter but you do not know that the counterfeit coins are of identical weight. Find the strategy which identifies the counterfeit coins with a balance scale using at most three weighings.

Problem A4:
Find the positive integer A such that exactly two of the following statements are true:

(a) $A + 82$ is the square of an integer;

(b) the last digit of A is 5;

(c) $A - 7$ is the square of an integer.

[1]Denis Potapov is a Senior Lecturer in the School of Mathematics and Statistics at UNSW Sydney.

Problem A5:
Two players are given a piece of paper which has two integers 25 and 36 written on it. On every move, a player can add a positive integer to the paper subject to the following conditions:

(a) the added number is not equal to any of the existing numbers on the paper;

(b) the added number is the difference of any two different numbers already present on the paper.

The player who is unable to make a move loses.
Prove that the second player always wins and find the number of moves in the game.

Problem A6:
Two points A, and B are chosen in the plane.
Find the set of all points M such that the $AM : BM = 2 : 1$.

B Senior Division – Problems

Problem B1:
Let n be a positive integer. Prove that the product of all of the positive divisors of n is equal to $\sqrt{n^s}$, where s is the number of divisors of the integer n.

Problem B2:
Is it possible to draw five straight lines on the plane such that every line intersects exactly three other lines?

Problem B3:
A notebook has 40 statements. The first statement reads "This book has one and exactly one false statement"; the second statement reads "this book has two and exactly two false statements"; and so forth, with the last statement reading "this book has forty and exactly forty false statements".

Which of the statements above are true?

Problem B4:
You are given six coins and you know that two of them are counterfeit. You also know that every counterfeit coin is 0.1g lighter than every genuine coin. You have a balance scale but the scale is unable to register less than 0.2g difference in weight. Find the strategy which identifies the counterfeit coins and which uses the balance scale at most four times.

Problem B5:
Two players, Alice and Bob are given a piece of paper with the number 1023 written on it. Each player makes a move by writing a smaller integer which is greater or equal to half of the preceding number. The game ends when the number 1 is written. The player who writes 1 is the winner. Find the winner and a winning strategy.

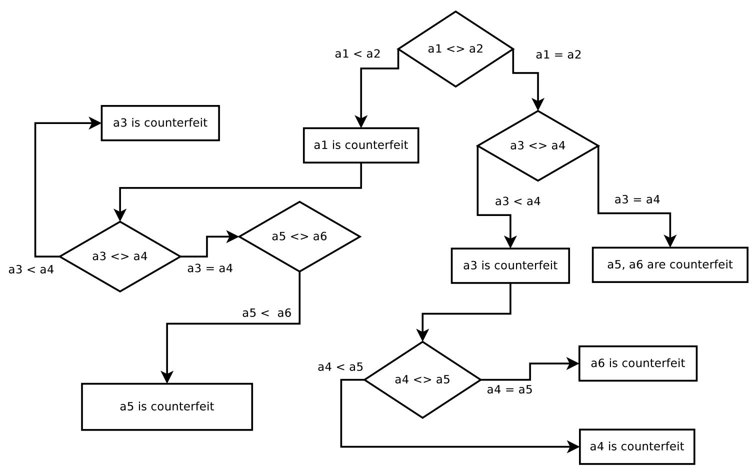
Problem B6:
Let $\triangle ABC$ be a triangle and $\mathcal{S}$ be the corresponding circumscribed circle. Let Q be a point on $\mathcal{S}$. Prove that the bases of perpendiculars dropped from Q to the sides of the triangle $\triangle ABC$ lie on a straight line.

A Junior Division – Solutions

Solution A1.
Any divisor $d < \sqrt{n}$ of an integer n corresponds to the divisor $n/d > \sqrt{n}$. Hence, an integer has en even number of divisors unless $\sqrt{n}$ is an integer. $\square$

Solution A2.
Answer: *No.*
Assume that the answer is yes, and consider the ordered pairs (ℓ_1, ℓ_2) of line segments ℓ_1 and ℓ_2 that intersect. There are $3 \times 5 = 15$ such pairs since each of the five line segments ℓ_1 intersect exactly three other line segments ℓ_2. However, whenever (ℓ_1, ℓ_2) is a pair of line segments ℓ_1 and ℓ_2 that intersect, then so is (ℓ_2, ℓ_1), so the number of these pairs must be even, a contradiction. $\square$

Solution A3.
Answer: A brief solution is explained by the following diagram:



Solution A4.
Answer: 1943. The last digit of a square of an integer is either

$$0, \quad 1, \quad 4, \quad 5, \quad 6 \quad \text{or} \quad 9\,.$$

Hence, if the second statement is true, then the first statement is false (the last digit of $A + 82$ is 7); and the last statement is false (the last digit of $A - 7$ is 8). Therefore, the second statement is false and the other two are true.

Assume that

$$A + 82 = p^2 \quad \text{and} \quad A - 7 = q^2\,.$$

We then have

$$(p-q)(p+q) = p^2 - q^2 = 82 + 7 = 89\,.$$

Since 89 is prime,

$$p - q = 1 \quad \text{and} \quad p + q = 89$$

or

$$p = 45 \quad \text{and} \quad q = 44\,.$$

Solution A5.

Only the numbers less than or equal to 36 of the following format can appear on the paper

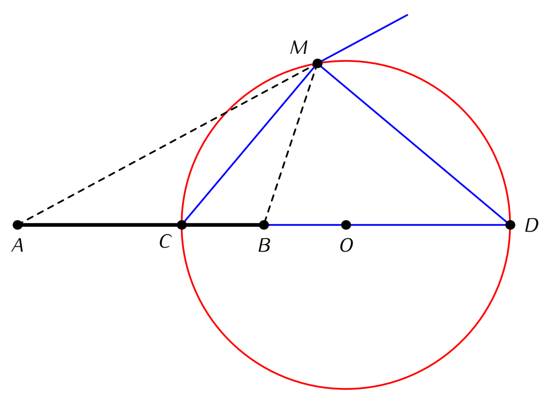
$$25m + 36n, \quad m, n \in \mathbb{Z}\,.$$

Hence, $\gcd(36, 25)$ will appear on the paper at some stage; hence, every positive multiple of $\gcd(36, 25)$ which is less or equal to 36 will appear on the paper.

Since $\gcd(36, 25) = 1$, every integer between 1 and 36 will appear on the paper eventually. That is, the rule of the game allows 34 moves all together, and the second player is the last to make the move. $\square$

Solution A6.

Answer: The set is a circle with centre on the line AB.



The solution below works for $AM : BM = a : 1$. Let MC be the bisector of the angle $\angle AMB$ and C be the point on AB, and let MD be the bisector of the angle between the extension of AM beyond M and MB. The angle $\angle CMD$ is $90^{\mathcal{S}}$.

Let us show that the intersection points C and D are independent of the point M. By using the *Law of Sines* applied to the triangle $\triangle AMC$ and using the fact that

$$\frac{AM}{BM} = a \quad \text{and} \quad \angle AMC = \angle BMC$$

we see that

$$\frac{AC}{BC} = a \, .$$

Similarly, by using the *Law of Sines* on the triangles $\triangle BMD$ and $\triangle AMD$, and using the fact that

$$\angle AMD + \angle BMD = 180^{S} \, ,$$

we see that

$$\frac{AD}{BD} = a \, .$$

B Senior Division – Solutions

Solution B1.

If $p_1, p_2, \ldots, p_s$ is the list of divisors in ascending order and $q_1, q_2, \ldots, q_s$ is the list of the same divisors in descending order, then

$$n^s = (p_1 q_1) \times (p_2 q_2) \times \ldots \times (p_s q_s) \, . \qquad \square$$

Solution B2.

Answer: *No*.

Assume that the answer is yes, and consider the ordered pairs (ℓ_1, ℓ_2) of line segments ℓ_1 and ℓ_2 that intersect. There are $3 \times 5 = 15$ such pairs since each of the five line segments ℓ_1 intersect exactly three other line segments ℓ_2. However, whenever (ℓ_1, ℓ_2) is a pair of line segments ℓ_1 and ℓ_2 that intersect, then so is (ℓ_2, ℓ_1), so the number of these pairs must be even, a contradiction. $\square$

Solution B3.

The only true statement is the one before the last one.

Solution B4.

Assume that the coins are indexed $1, 2, \ldots, 6$. The first two weighings are

$$1,2,3 \quad [A] \quad 4,5,6 \quad \text{and} \quad 1,2,4 \quad [B] \quad 3,5,6 \, ,$$

where each relation [A] and [B] is either $[<], [>], [=]$. Consider all possible outcomes.

In the case that $[A] = [<]$ and $[B] = [<]$, the counterfeit coins are $1, 2$.

The other three cases in which both [A] and [B] register weight difference are similar.

In the case that $[A] = [<]$ and $[B] = [=]$, the counterfeit coins are either $1, 3$ or $2, 3$. To find out which of the latter is the counterfeit pair, we use another two weighings:

$$1, 3 \quad [C] \quad 4, 5 \quad \text{and} \quad 2, 3 \quad [D] \quad [4, 5]$$

with the known genuine coins $4, 5$. The other three cases when one of the weighings $[A]$ or $[B]$ registers difference are similar.

In the case that $[A] = [=]$ and $[B] = [=]$, every possible pair which appeared on one side in one of the weighings [A] or [B] is *not* a pair of two counterfeit coins. We cross out every such pair in the table below.

	1	2	3	4	5	6
1	×	×	×	×	·	·
2		×	×	×	·	·
3			×	·	×	×
4				×	×	×
5					×	×
6						×

Hence, the possible counterfeit pairs are

$$(1, 5), (1, 6), (2, 5), (2, 6), (3, 4) .$$

Weigh the pairs

$$1, 5 \quad [C] \quad 2, 6 \quad \text{and} \quad 1, 6 \quad [D] \quad 2, 5 .$$

If $[C] = [<]$, then $1, 5$ is the counterfeit pair. The other three cases in which either [C] or [D] register difference are similar.

Finally, if $[C] = [=]$ and $[D] = [=]$, then the counterfeit is $(3, 4)$. □

Solution B5.

Consider the sequence

$$a_1 = 1 \quad \text{and} \quad a_k = 2a_{k-1} + 1, \quad k > 1 .$$

That is,

$$a_k = 2^k - 1 .$$

Note that $a_{10} = 1023$.

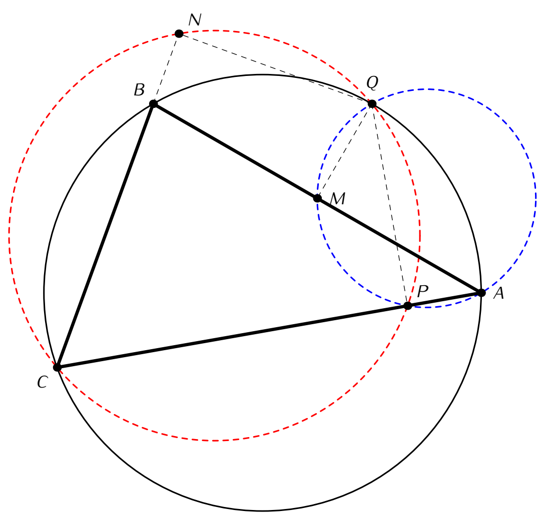
Whoever first writes an integer from this sequence is the guaranteed winner. Indeed, if an integer a_k is written, then the next move allows the player to write an integer from the range $[a_{k-1}+1, a_k-1]$ only. Hence, the following move allows the number a_{k-1} to be written and so on, leading such player to the winning move $a_1 = 1$.

If Alice plays first and the game starts with $a_{10} = 1023$, then Bob is the winner. $\square$

Solution B6.

Let M, N, P be the bases of the perpendiculars as shown on the diagram below.

Note first that the triangles $\triangle PQA$ and $\triangle QMA$ are right triangles and share their hypotenuse. That is, the quadrangle $QAPM$ is inscribed. The quadrangle $NQPC$ is inscribed for a similar reason.



Let δ_B and δ_R be the corresponding circles (blue and red) on the diagram below, and let us show that

$$\angle QPM = \angle QPN\,.$$

Note that $\angle QPM = \angle QAM$ since the both angles rest on the same arc of the circle δ_B. Furthermore, $\angle MAQ = \angle BAQ = \angle BCQ$ since the last two rest on the same arc of the circle $\mathcal{S}$. Finally, $\angle NCQ = \angle NPQ$ since the latter two rest on the same arc of the circle δ_R.