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56th UNSW School Mathematics Competition

Solutions by Denis Potapov[1]

A Junior Division – Problems

Problem A1:
In the country *Digit-land*, there are nine cities: "1", "2", ..., "9".
Two cities A and B are connected by a flight if and only if the two-digit number AB is divisible by 3. Is there a way to travel by flights from the city "1" to the city "9"?

Problem A2:
A court hears the case of three suspects: Brown, Jones and Smith. One of them has committed a crime. Every suspect has made two statements, as follows.

Brown:	"I did not do it", "Smith did it";
Jones:	"Smith is innocent", "Brown did it";
Smith:	"I did not do it", "Jones did not do it".

The court has established that

(a) one of the suspects lied two times;

(b) one of the suspects told the truth two times;

(c) one of the suspects lied once and told the truth once.

Find out who has committed the crime.

Problem A3:
Show that the digits of a six-digit number can always be ordered so that the difference between the sum of the first three digits and the sum of the remaining three digits is less than 10.

Problem A4:
A plane is covered by an infinite square grid. Every square of the grid is painted by one of six colours. Prove that there are four squares of the grid painted with the same colour such that the centres of these squares form the corners of a rectangle with sides parallel to the lines of the grid. □

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Problem A5:
Two straight lines pass through two vertices of a triangle such that the triangle is cut into four smaller pieces: three triangles and a quadrilateral. It is possible to choose the lines such that the areas of these pieces are the same?

Problem A6:
There is a wolf at the centre of a square block of land. There is a dog located at each of the four vertices of the square. The wolf is allowed to move freely within the square and the dogs are only allowed to run by the sides of the square. Every dog is 50% faster than the wolf. A dog alone is unable to stop the wolf. On the other hand, the wolf cannot pass if met by any two dogs. Find the strategy for the dogs to ensure that the wolf does not escape the square.

B Senior Division – Problems

Problem B1:
A plane is covered by an infinite square grid. Every square of the grid is painted by one of six colours. Prove that there are four squares of the grid painted with the same colour such that the centres of these squares make a rectangle with sides parallel to the lines of the grid. □

Problem B2:
Find two two-digit numbers a and b such that the four-digit number ab is divisible by the product $a \times b$.

Problem B3:
Two friends, Alice and Bob, have a weird habit: Alice tells lies on Tuesdays, Thursdays and Saturdays; and Bob tells lies on Mondays, Tuesdays and Wednesdays. The friends tell the truth on every other day. Once, one of the friends was asked:

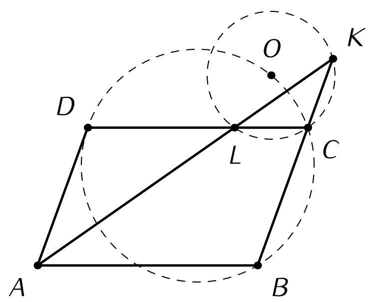
> "What is your name?"
> "Bob." — he replied.
> "Which day is today?"
> "Yesterday was Sunday."
> "And tomorrow will be Friday." — the second friend added.
> "Are you telling the truth?" — the second friend was asked.
> "I always tell the truth on Wednesdays." — the second friend replied.

Identify the names of both friends in this conversation; also,
identify the day of the week of this conversation.

Problem B4:
Every point of a plane is painted with one of two different colours such that every equilateral triangle of side 1 has two of its vertices painted with different colours. Prove that there is an equilateral triangle with side $\sqrt{3}$ such that its every vertex is painted with the same colour.

Problem B5:
There is a rabbit in the centre of a square block of land. There is a wolf located at each of the four vertices of the square. The rabbit is allowed to run freely within the square and the wolves are only allowed to run by the sides. Every wolf is 40% faster than the rabbit. Is there a strategy for the rabbit to escape the square?

Problem B6:
Let $ABCD$ be a parallelogram and let the bi-sector of the angle $\angle BAD$ intersect the side BC and the side CD in the points K and L, respectively. Prove that the centre of the circle through the points C, K and L lies on the circle through the points B, C and D.



A Junior Division – Solutions

Solution A1.
Answer: *No*. Assume that there is a way to travel from "1" to "9". Let $x_1, x_2, \ldots, x_n$ be the sequence of stop-overs. It follows from the divisibility by 3 test that

$$1 + x_1 \equiv 0 \pmod 3, \quad x_1 + x_2 \equiv 0 \pmod 3, \quad \ldots, \quad x_n + 9 \equiv 0 \pmod 3.$$

Therefore,

$$x_1 \equiv -1 \pmod 3, \quad x_2 \equiv 1 \pmod 3, \quad \ldots, \quad 9 \equiv \pm 1 \pmod 3.$$

The latter is false and hence the initial assumption is incorrect. □

Solution A2.
Answer: *Smith*. If it were *Brown* who committed the crime, then the true/false table of the statements would be as follows:

B1	B2	F	F
J1	J2	T	T
S1	S2	T	T

If it were *Jones* who committed the crime, then the true/false table of the statements would be as follows:

B1	B2	T	F
J1	J2	T	F
S1	S2	T	F

If it were *Smith* who committed the crime, then the true/false table of the statements would be as follows:

B1	B2	T	T
J1	J2	F	F
S1	S2	F	T

□

Solution A3.
Let us order the digits so that

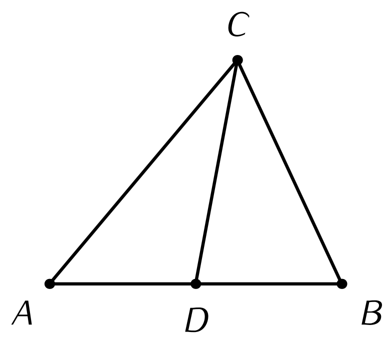
$$a_1 \geq a_4 \geq a_2 \geq a_5 \geq a_3 \geq a_6 .$$

Then

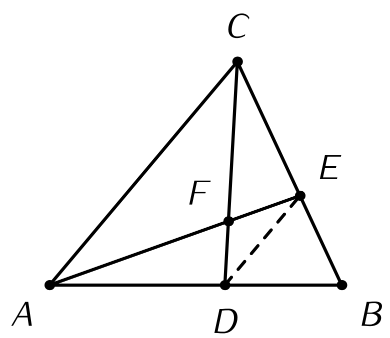
$$(a_1 + a_2 + a_3) - (a_4 + a_5 + a_6) \leq (a_1 + a_2 + a_3) - (a_2 + a_3 + a_6) = a_1 - a_6 \leq 9 . \qquad \square$$

Solution A4.
Choose any horizontal seven-squares-high strip on the plane. Note that there are finitely many different ways to paint squares of a one-square-wide and seven-squares-high column. Hence, there are two such columns in the strip we chose earlier which are painted with identical sets of colours. Since, we only have six colours and the columns have seven squares, there will be two squares in each column painted with identical colours. These squares make the rectangle as required. □

Solution A5.
Answer: *No*. The proof is based on the following observation (see picture). If the triangle $\triangle ABC$ is split by the segment CD into pieces of equal area, then $AD = BD$.

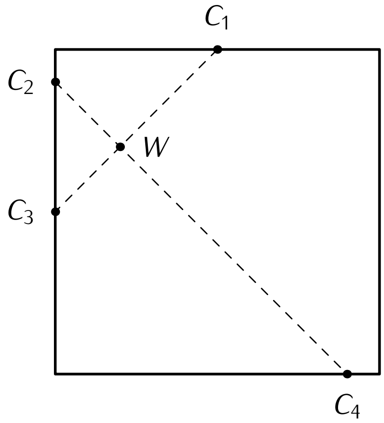


Assume that the areas of triangles $\triangle AFD$, $\triangle AFC$ and $\triangle CFE$ (see picture) are equal. It then follows from the observation above that $AF = FE$ and $CF = FD$. That is, the quadrilateral $ADEC$ is a parallelogram. The latter is false since the sides AD and CE intersect.



□

Solution A6.
Run two lines through the position of the wolf parallel to each of the diagonals of the square. Let C_1, C_2, C_3, C_4 be the points of intersection of these lines with the sides of the square. If v is the speed of the wolf, then every point $C_1, \ldots, C_4$ moves with the speed at most $v\sqrt{2} < \frac{3}{2}v$. Hence, if the dogs stick to the points $C_1, \ldots, C_4$, then the wolf will always be met by two dogs at the time it arrives to the square boundary.



□

B Senior Division – Solutions

Solution B1.
Choose any horizontal seven-squares-high strip on the plane. Note that there are finitely many different ways to paint squares of a one-square-wide and seven-squares-high column. Hence, there are two such columns in the strip we chose earlier which are painted with identical sets of colours. Since, we only have six colours and the columns have seven squares, there will be two squares in each column painted with identical colours. These squares make the rectangle as required. □

Solution B2.
Answer: 17 and 34; or 13 and 52.

We have $100a + b = kab$, so $b = a \times (kb - 100)$. Hence, a divides b. Write $b = ma$. We then have $100 = m(ka - 1)$. That is, m is a divisor of 100 and, since $m = \frac{b}{a}$, it follows that $1 \leq m \leq 10$. Therefore, the possible values of m are 1, 2, 4, and 5. We now look for a among two-digit divisors of $\frac{100}{m} + 1$:

$m = 1$	no divisors
$m = 2$	$a = 17, 51$
$m = 4$	$a = 13, 26$
$m = 5$	no divisors

□

Solution B3.
Let the first friend in the conversation be F1 and the second friend be F2.

If F1 is "Bob", then he tells the truth in his first reply. In his second reply, he says that the day of the week is "Monday". Since Bob is telling the truth, the second reply must be true. The latter contradicts the fact that Bob is lying on Mondays. Hence, the initial assumption that F1 is "Bob" is incorrect.

Hence, F1 is "Alice" and F2 is "Bob".

Note that F1/Alice is lying in the conversation and Alice is lying on Tuesdays, Thursdays and Saturdays.

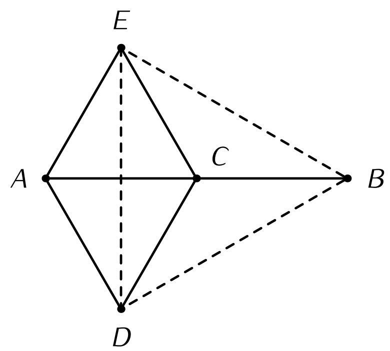
F2/Bob's second reply is a lie; hence F2/Bob is lying and Bob tells lies on Mondays, Tuesdays and Wednesdays.

The common day is *Tuesday*. □

Solution B4.
Let us show first that there is a length 2 straight line segment AB with end points painted with different colours.

Assume that the claim is not true. It then follows that every point on every circle of radius 2 is painted the same colour as the colour of the centre of such circle.

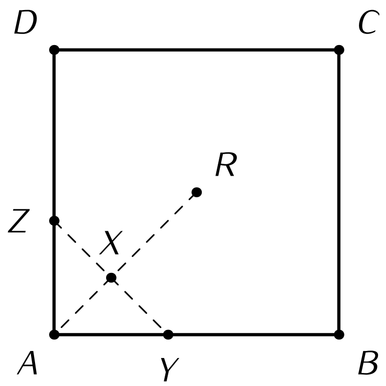
Fix one such circle γ. Consider every circle of radius 2 with centre on γ. Every such circle is painted with one colour and this colour is the same as the colour of γ. Hence, the entire disk of radius 4 co-centred with γ is painted with the same one colour. Within such disk there is an equilateral triangle with side 1 whose vertices are painted with one colour. This contradicts the statement of the problem. Hence, the initial assumption is incorrect; therefore, there is a length 2 straight light segment AB with differently coloured end points.



Let C be the midpoint of AB and let assume without loss of generality that C is painted with the same colour as A. Let us construct two equilateral triangles $\triangle ACE$ and $\triangle ACD$ on both sides of the line AB (see picture). Since A and C are identically coloured, the points E and D are identically coloured and the latter colour is the same as B. Hence, the triangle $\triangle BDE$ satisfies the requirements of the problem. □

Solution B5.

Answer: *Yes*. Consider the following strategy (see picture). The rabbit starts from the centre R towards the vertex A. Let us assume that when the rabbit arrives to the point X (to be identified later), the wolf A is on the side AD. In such case, the rabbit turns $90°$ to the point Y (in case the wolf A is on the side AB, the rabbit turns to the point Z).



Let $AX = x$ and let v_R and v_W be the maximal speeds of the rabbit and the wolves, respectively. The shortest arrival times of the rabbit and the wolf B to the point Y are as follows (assuming that the side of the square is 1):

$$\frac{1}{v_R\sqrt{2}} \quad \text{and} \quad \frac{(1 - x\sqrt{2})}{v_W}.$$

Hence, the condition of escape from wolf B is that $\frac{1}{v_R\sqrt{2}} < \frac{(1-x\sqrt{2})}{v_W}$, which means that

$$0 < x < \frac{1}{2}\left(\sqrt{2} - \frac{v_W}{v_R}\right).$$

If we mark the time since the moment the rabbit passed the point X, then the shortest arrival times of the rabbit and wolf A to the point Y are as follows:

$$\frac{x}{v_R} \quad \text{and} \quad \frac{x\sqrt{2}}{v_W}.$$

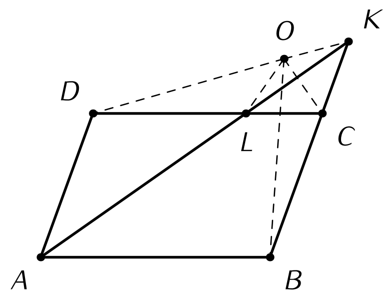
Hence, the condition of escape from wolf A is that $\frac{x}{v_R} < \frac{x\sqrt{2}}{v_W}$, implying that $\frac{v_W}{v_R} < \sqrt{2}$. Since $\frac{v_W}{v_R} = 1.4 < \sqrt{2}$, both conditions above can be met and a suitable choice x can be made. $\square$

Solution B6.
The latter is sufficient to claim that To prove that the quadrilateral $BCOD$ is circumscribed, we just need to prove that $\angle DOB = \angle DCB$.

Since AK is a bi-sector, it follows that $\angle DAL = \angle BAL$. Furthermore, we also have $\angle KLC = \angle LKC = \angle KAB$. Consequently, $LC = CK$ and $BK = AB$. Now, $\triangle OCL = \triangle OKC$ since the corresponding sides are equal. Hence, $\angle KOC = \angle LOC$.

If we rotate the segment KB around point O by the angle $\angle KOC$, then the point K rotates to the point C; the point C rotates to the point L. Hence, $\angle DCB$ is the angle between LC and KC, namely the angle of rotation.

On the other hand, since $BK = AB = CD$, the point B rotates to the point D. Hence, $\angle DOB$ is the angle of rotation. That is, $\angle DOB = \angle DCB$.



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