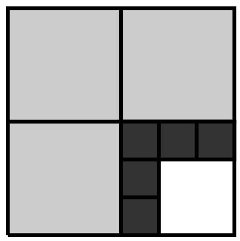
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 18, September 15, 2015[1]

1. Yes; see figure below.



2. We need k and n such that, $2^{k+1}+\ldots+2^{k+10} = 1+2+\ldots+n$. Simplifying $2^{k+2}(2^{10}-1) = n(n+1)$, which holds for $n = 2^{10}-1$ and $k = 8$.

3. Construct a directed graph with 2000 vertices representing the people, with directed edges connecting the vertices. We want to find the smallest number of vertices with two edges connecting them. Each vertex has 1000 outgoing edges, so there is a total of 1000×2000 edges. The number of ways we can pair two vertices together is $2000 \times 1999 \div 2 = 1999 \times 1000$. So in the best scenario, whereby each vertex of the graph is connected by at least one edge, we still have $2000 \times 1000 - 1999 \times 1000 = 1000$ edges left over. The extra edges each have to connect vertices that are already connected, hence giving us at least 1000 pairs of friends.

4. Three. The Joker can look at the first three digits of Batman's password and copy them down, then he can then use the fact that the passwords are unique, and deduce the other three digits by looking at the last three digits of all other clients.

5. Let the total number of students be T, the number of those who knows all three languages a, the number of those who knows English and French only x, the number of those who knows English and Germany only y, and the number of those who knows Germany and French only z. So we want to show that $\frac{a}{a+z} > 0.9$.

   We are given $\frac{a+y}{T} \geq 0.9$ and $\frac{a+x}{T} \geq 0.9$. Since $\frac{a+y}{a+x+y+z} \geq \frac{a+x}{T} > 0.9$, we have $a+y > 9(x+z)$. Similarly, $a+x > 9(y+z)$. It follows that $2a+9(x+y) > (a+x)+(a+y) > 9(x+z)+9(y+z) = 9(x+y)+18z$. Hence $a > 9z$ and $10a > 9(a+z)$, so that $\frac{a}{a+z} > \frac{9}{10}$.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

6.

Senior Questions

1. Let
$$a_n = \sqrt[n]{\frac{(2n)!}{n!n^n}}.$$
If we take the natural log on both sides of the above equation, then
$$\begin{aligned}\ln(a_n) &= \frac{1}{n}\left(\ln\left(\frac{(n+1)\times(n+2)\times\ldots\times(n+n)}{n^n}\right)\right)\\ &= \frac{1}{n}\left(\ln\left(\frac{n+1}{n}\right)+\ln\left(\frac{n+2}{n}\right)+\ldots+\ln\left(\frac{n+n}{n}\right)\right)\\ &= \sum_{k=1}^{n}\frac{1}{n}\ln\left(1+\frac{k}{n}\right).\end{aligned}$$
Notice that the RHS of the last displayed equation is a Riemann sum of $\ln(x)$ for $x\in[1,2]$. As n gets very large, the Riemann sum for the approximation of $\ln(x)$ becomes the exact integral
$$\int_1^2 \ln(x)\,dx = 2\ln 2 - 1,$$
where we have solved the integral using integration by parts. Hence $\lim_{n\to\infty} a_n = \exp(2\ln 2 - 1)$ (to obtain the conclusion, we have used the fact that $\ln(\lim_{n\to\infty} a_n) = \lim_{n\to\infty}\ln a_n$, why is this true?).

2. Lets label the two known equations:
$$x^4y^5 + y^4x^5 = 810, \tag{1}$$
and
$$x^3y^6 + y^3x^6 = 945. \tag{2}$$
Consider $(x+y)^3 = x^3+y^3+3(x^2y+y^2x)$. From (1), we have $x^2y+y^2x = \frac{810}{(xy)^3}$ and $x+y=\frac{810}{(xy)^4}$. Therefore
$$\begin{aligned}(x+y)^3 &= x^3+y^3+3(x^2y+y^2x)\\ \frac{810^3}{(xy)^{12}} &= x^3+y^3+\frac{2430}{(xy)^3}.\end{aligned}\tag{3}$$
Also, from (2) $x^3+y^3=\frac{945}{(xy)^3}$, therefore the second line of (3) becomes
$$\frac{810^3}{(xy)^{12}} = \frac{945}{(xy)^3}+\frac{2430}{(xy)^3},$$
which implies $(xy)^9 = \frac{810^3}{945+2430}$. Therefore, $(xy)^3 = 54$ and $x^3+y^3 = 17.5$.
We can now easily evaluate $2x^3+(xy)^3+2y^3 = 89$.

3. Consider the finite sum $S_n = \prod_{k=2}^{n} \cos\left(\frac{\pi}{2^k}\right)$. Multiplying S_n by $\sin\left(\frac{\pi}{2^n}\right)$, then using the trigonometry identity $2\sin x \cos x = \sin 2x$ yields

$$\begin{aligned}
\sin\left(\frac{\pi}{2^n}\right) \times S_n &= \sin\left(\frac{\pi}{2^n}\right)\cos\left(\frac{\pi}{2^n}\right) \times S_{n-1} \\
&= \frac{1}{2}\sin\left(\frac{\pi}{2^{n-1}}\right) \times S_{n-1} \\
&= \dots \\
&= \frac{\sin\left(\frac{\pi}{2}\right)}{2^{n-1}} = \frac{2}{2^n}.
\end{aligned}$$

Therefore, $S_n = \frac{2}{2^n} \div \sin\left(\frac{\pi}{2^n}\right) = \frac{2}{\pi}\left(\frac{\pi}{2^n} \div \sin\left(\frac{\pi}{2^n}\right)\right)$. But

$$\begin{aligned}
\lim_{n\to\infty} \frac{\pi/2}{\sin(\pi/2)} &= \lim_{x\to 0} \frac{x}{\sin x} \\
&= 1,
\end{aligned}$$

where equality on the second line is due to the L'Hospital rule. Thus we conclude that $\lim_{n\to\infty} S_n = \frac{2}{\pi}$.