

MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 18, September 15, 2015¹

1. Is it possible to cut a square into nine squares and colour one of them white, three of them gray and five of the black, such that squares of the same colour have the same size and squares of different colours will have different size?
2. Billy summed up 10 consecutive powers of 2, starting from some power, while Joe summed up several consecutive positive integers starting from 1. Can they get the same results?
3. A new website registered 2000 people. Each of them invited 1000 other registered people to be their friends. Two people are considered to be friends if and only they have invited each other. What is the minimum number of pairs of friends on this website?
4. A bank has one million clients, one of whom is Batman. Each client has a unique PIN number consisting of six digits. The Joker has a list of all the clients. He is able to break into the account of any client, choose any n digits of the PIN number and copy them. The n digits he copies from different clients need not be in the same n positions. He can break into the account of each client, but only once. What is the smallest value of n which allows The Joker to determine the complete PIN number of Batman?
5. In a school, more than 90% of the students know both English and German, and more than 90% of the students know both English and French. Prove that more than 90% of the students who know both German and French also know English.
6. An astronaut plants a flagpole on the surface of the Moon until the top of the flagpole is at eye-level height, at 1.5 metres, and then the astronaut walks off over level ground until the top of the flagpole is just visible on the horizon. The astronaut uses a laser distance meter to measure the straightline distance from his line of sight to the top of the flagpole at 4.6 kilometres. What is the radius of the Moon?

¹Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

Senior Questions

1. Evaluate

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n)!}{n!n^n}}.$$

Hint: use calculus.

2. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.
3. Let the notation $\prod_{i=1}^k a_i$ denotes the product of a_i for $i = 1, 2, \dots, k$. For example, if $a_i = i$ then $\prod_{i=1}^k a_i = k!$. Find

$$\prod_{k=2}^{\infty} \cos\left(\frac{\pi}{2^k}\right).$$